A General Fredholm Theory II: Implicit Function Theorems

by

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1 Introduction

This paper is a continuation of [12]. There we introduced a new concept of smoothness in infinite dimensional spaces extending the familiar finite-dimensional smoothness concept. This allowed us to introduce the concept of a splicing leading to new local models for smooth spaces. These local models can be of finite or infinite dimension and it is an interesting feature that the local dimensions need not to be constant. In [12] we defined smooth maps between these local models and constructed a tangent functor. Applying classical constructions from differential geometry to these new objects we constructed a generalized differential geometry, the so-called **splicing based differential geometry**. The generalization of an orbifold to the splicing world leads to the notion of a polyfold and the generalization of a manifold to the notion of a M-polyfold. Not surprisingly the generality of the new theory allows constructions which do not parallel any classical construction. Benefits of this new differential geometric world include new types of spaces needed for an abstract and efficient treatment of theories like Floer-Theory, Gromov-Witten or Symplectic Field Theory from a common point of view. From our point of view the afore-mentioned theories are built on a suitable counting of zeros of a smooth Fredholm section of a strong polyfold bundle. The fact that the Fredholm sections on different connected components are related in subtle ways leads to interesting algebraic structures which can

be captured on an abstract level, see [17]. An abstract Sard-Smale-type perturbation theory takes care of the usual intrinsic transversality difficulties well-known in al these theories.

The implicit function theorem and Sard-type transversality theorems are crucial analytical tools in differential geometry and nonlinear analysis (where one usually refers to a nonlinear Fredholm theory and a Sard-Smale-type transversality theory). In the present paper we develop the analogous 'splicing based' theory. We carry out the analysis needed for a generalized Fredholm theory for sections of strong M-polyfold bundles. The analytical results are a crucial technical tool in the theory of Fredholm functors and polyfolds presented in the forthcoming part III in [13] which is needed for symplectic field theory treated in [16].

Before we describe the contents of the present paper we should briefly mention the prerequisites, namely the material introduced in part I of this series in [12]. For the convenience of the reader a glossary about the new concepts is added in section 7. In there we recall, in particular, the notion of sc-smoothness, the notions of an M-polyfold and of a strong bundle splicing. Moreover, the definitions of an sc^+-section of a strong M-polyfold bundle and of a linearized section are recalled. A more comprehensive treatment can be found in [10].

The main technical results concern the very special sc-smooth maps called contraction germs (Definition 2.1). They arise in the normal forms for Fredholm sections introduced in Definition 3.6. Contraction germs allow an 'infinitesimal smooth implicit function theorem' (this extremely local version of an implicit function theorem is a special feature of the sc-world). The difficult part is the regularity of the solution, while the existence of a continuous solution germ is an immediate consequence of the contraction principle in complete spaces (see Theorems 2.3 and 2.6). Armed with the notion of a Fredholm section and the infinitesimal implicit function theorem we derive more familiar versions of the implicit function theorem. Of particular interest are the Theorems 4.6 and 4.18 which describe the local structure of the solution set of Fredholm sections. Based on these results we then develop in section 5 the global Fredholm theory. It consists of a transversality and perturbation theory. Two results of particular interest are the Theorems 5.21 and 5.22. These results look like those familiar from classical Fredholm theory. The reader should, however, keep in mind that the ambient spaces are M-polyfolds which are much more general spaces than Banach manifolds and that the notion of a Fredholm section is much more general as well. Here

is, as a sample, a version of Theorem 5.22.

Theorem. *Let $p : Y \to X$ be a fillable strong M-polyfold bundle and f a proper Fredholm section. Then given an open neighborhood U of $f^{-1}(0)$ there exist arbitrarily small sc$^+$-sections s supported in U having the property that the Fredholm section $f + s$ is in general position to the boundary ∂X (as defined in Definition 5.17) so that the solution set $(f + s)^{-1}(0)$ is a smooth compact manifold with boundary with corners contained in U.*

If s_0 and s_1 we are two such sc$^+$-sections which are sufficiently small (Theorem 5.22 formulates this quantitatively) and $\mathcal{M}_i = (f + s_i)^{-1}(0))$ are associated solutions sets, then we can find a smooth arc s_t for $t \in [0, 1]$ of sc$^+$-sections connecting s_0 with s_1 so that

$$(t, x) \mapsto f(x) + s_t(x)$$

has a regular compact solution set $\mathcal{M} = \{(t, x) \mid f(x) + s_t(x) = 0\}$ which lies in general position to the boundary of $[0, 1] \times X$. In particular, $\mathcal{M}$ is a manifold with boundary with corners and moreover the subsets of points (t, x) in $\mathcal{M}$ with $t = 0$ or $t = 1$ are the solution sets $\mathcal{M}_i$ for $i = 0, 1$.

Under the assumptions of our theorem, we conclude in the case $\partial X = \emptyset$ that the (un-oriented) cobordism class of the solution set is an invariant. If we are dealing with oriented Fredholm sections we arrive at invariants in the oriented cobordism category.

In order to describe another sample result we make use of the notion of an auxiliary norm which will be explained in chapter 5. It allows to measure the size of perturbations. We shall also introduce later on in section 5.5 our version of the de Rham cohomology $H^*_{\mathrm{dR}}(X, \mathbb{R})$ in the sc-smooth setting.

Theorem (Invariants). *Let $p : Y \to X$ be a fillable strong M-polyfold bundle, where the M-polyfold X has no boundary. We assume that f is a proper oriented Fredholm section. We suppose further that N is an auxiliary norm and U an open neighborhood of the solution set $f^{-1}(0)$, so that (N, U) controls compactness. Then there is a well-defined map*

$$\Phi_f : H^*_{dR}(X, \mathbb{R}) \to \mathbb{R}$$

satisfying

$$\Phi_f([\omega]) = \int_{\mathcal{M}^{f+s}} \omega$$

for every generic solution set $\mathcal{M}^{f+s} = (f+s)^{-1}(0)$, where s is an sc^+-section having support in U and satisfying $N(s) < 1$. Moreover, if $t \to f_t$ is a proper homotopy of oriented Fredholm sections then

$$\Phi_{f_0} = \Phi_{f_1}.$$

There are also results for M-polyfolds with boundaries ∂X involving the relative de Rham cohomology $H^*_{dR}(X, \partial X)$. We have proved such generalizations for Fredholm sections of strong polyfold bundles in [13] and will not address this topic here.

As already mentioned, the traditional concept of nonlinear Fredholm maps is not sufficient for our purposes. In order to motivate our new definition of a Fredholm section of sc-vector bundles we first recall from [1] the classical local analysis of a smooth map $f : E \to F$ between Banach spaces defined near the origin in E and satisfying $f(0) = 0$. We assume that the derivative $Df(0) \in \mathcal{L}(E, F)$ is a Fredholm operator. Consequently, there are the topological splittings

$$E = K \oplus X \quad \text{and} \quad F = C \oplus Y$$

where $K = \ker Df(0)$ and $Y = \operatorname{im} Df(0)$ and $\dim K < \infty$, $\dim C < \infty$. In particular,

$$Df(0) : X \to Y$$

is a linear isomorphism of Banach spaces. It allows to introduce the linear isomorphism $\sigma : K \oplus X \to K \oplus Y$ defined by $\sigma(k, x) = (k, Df(0)x)$. In the new coordinates of the domain, the map f becomes the composition $h = f \circ \sigma^{-1} : K \oplus Y \to C \oplus Y$, satisfying $h(0) = 0$ and

$$Dh(0)[k, y] = [0, y].$$

If we write $h = (h_1, h_2)$ according to the splitting of the target space and denote by $P : C \oplus Y \to Y$ the canonical projection, then the mapping $h_2(k, y) = P \circ f \circ \sigma^{-1}(k, y)$ has the following normal form

$$h_2(k, y) = y + B(k, y),$$

where $B(0) = 0$ and $DB(0) = 0$. Hence the maps

$$y \mapsto B(k, y)$$

are contractions near the origin in Y having arbitrary small contraction constants $0 < \varepsilon < 1$ if k and y are sufficiently close to 0 depending on the contraction constant. In other words, taking a suitable coordinate representation of f and a suitable projection onto a subspace of finite codimension in the target space, the non linear Fredholm map looks locally near 0 like a parametrized contraction perturbation of of the identity mapping.

In the setting of splicing based differential geometry one cannot rely anymore on the classical implicit function theorem. This classical theorem gives an insight into the behavior near a point at which only the linearization is known. In our approach to Fredholm sections we shall instead start from the contraction normal form and shall call a section Fredholm, if in appropriate coordinates it has a contraction normal form of the type above. Consequently, we begin the analysis with an implicit function theorem for contraction germs.

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2 Analysis of Contraction Germs

In this section we study the existence and the regularity of solution germs of special equations called contraction germs. The results form the analytical back bone for our implicit function theorems and Fredholm theory.

2.1 Contraction Germs

In [12] we have introduced an sc-structure on a Banach space E. It consists of a nested sequence

$$E = E_0 \supset E_1 \supset E_1 \supset \cdots \supset \bigcap_{m \geq 0} E_m =: E_\infty$$

of Banach spaces E_m having the properties that the inclusions $E_n \to E_m$ are compact operators if $m < n$ and the vector space E_∞ is dense in E_m for every $m \geq 0$. Points and sets contained in E_∞ are called **smooth points**

and **smooth sets**. A Banach space equipped with an sc-structure is called an sc-Banach space. We write E^m to emphasize that we are dealing with the Banach space E_m equipped with the sc-structure $(E^m)_k := E_{m+k}$ for $k \geq 0$.

A **partial quadrant** C in the sc-Banach space E is a closed subset which under a suitable sc-isomorphism $T : E \to \mathbb{R}^n \oplus W$ is mapped onto $[0, \infty)^n \oplus W$.

We denote by $\mathcal{O}(C, 0)$ an sc-germ of C-relative open neighborhoods of 0 consisting of a decreasing sequence

$$U_0 \supset U_1 \supset U_2 \supset \cdots \supset U_m \supset \cdots$$

of open neighborhoods U_m of 0 in $C_m = C \cap E_m$. The associated **tangent germ** $T\mathcal{O}(C, 0)$ of $\mathcal{O}(C, 0)$ consists of the decreasing sequence

$$U_1 \oplus E_0 \supset U_2 \oplus E_1 \supset \cdots \supset U_{m+1} \oplus E_m \supset \cdots .$$

Assume that F is a second Banach space equipped with the sc-structure (F_m). Then an $\mathbf{sc^0}$**-germ**

$$f : \mathcal{O}(C, 0) \to (F, 0)$$

is a continuous map $f : U_0 \to F_0$ satisfying $f(0) = 0$ and $f(U_m) \subset F_m$ such that

$$f : U_m \to F_m$$

is continuous for every $m \geq 0$. We extend the idea of an sc^1-map as defined in Definition 2.13 in [12] to the germ level as follows. An $\mathbf{sc^1}$**-germ** $f : \mathcal{O}(C, 0) \to (F, 0)$ is an sc^0-germ which is also of class sc^1 in the following sense. For every $x \in U_1$ there exists the linearisation $Df(x) \in \mathcal{L}(E_0, F_0)$ such that for all $h \in E_1$ with $x + h \in U_1$,

$$\frac{1}{\|h\|_1} \|f(x+h) - f(x) - Df(x)h\|_0 \to 0 \quad \text{as } \|h\|_1 \to 0.$$

Moreover, the tangent map $Tf : U_1 \oplus E_0 \to TF$, defined by

$$Tf(x, h) = (f(x), Df(x)h)$$

for $(x, h) \in U_1 \oplus E_0$, satisfies $Tf(U_{m+1} \oplus E_m) \subset F_{m+1} \oplus F_m$ and $Tf : T\mathcal{O}(C, 0) \to (TF, 0)$ is an sc^0-germ.

Definition 2.1. *Let E be an sc-Banach space and let V be a partial quadrant in a finite-dimensional vector space F. Then an sc^0-germ $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ is called an* $\mathbf{sc^0}$**-contraction germ** *if it has the form*

$$f(v, u) = u - B(v, u) \tag{1}$$

so that the following holds. For every level m and every $0 < \varepsilon < 1$ we have the estimate

$$\|B(v, u) - B(v, u')\|_m \leq \varepsilon \cdot \|u - u'\|_m$$

for all (v, u) and (v, u') close to $(0, 0)$ in $V \oplus E_m$. Here the notion of close depends on the level m and the contraction constant ε.

The following existence theorem is an immediate consequence of a parameter dependent version of Banach's fixed point theorem.

Theorem 2.2. *Let $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ be an sc^0-contraction germ. Then there exists a uniquely determined sc^0-germ $\delta : \mathcal{O}(V, 0) \to (E, 0)$ so that the associated graph germ $gr(\delta) : V \to V \oplus E$, defined by $v \mapsto (v, \delta(v))$, satisfies*

$$f \circ gr(\delta) = 0.$$

2.2 Regularity of Solution Germs

Our main concern now is the regularity of the unique solution δ of the equation $f(v, \delta(v)) = 0$ in Theorem 2.2. We shall prove that the solution germ δ is of class sc^k if the given germ f is of class sc^k. We first study the case $k = 1$ and abbreviate

$$V = [0, \infty)^l \times \mathbb{R}^{n-l} \subset \mathbb{R}^n.$$

Theorem 2.3. *If the sc^0-contraction germ $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ is of class sc^1, then the solution germ $\delta : \mathcal{O}(V, 0) \to (E, 0)$ in Theorem 2.2 is also of class sc^1.*

We are going to prove Theorem 2.3 under the following weaker assumptions on the sc^0-germ $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ having the above form $f(u, v) = u - B(v, u)$. We merely assume that for every level $m \geq 0$ there exists a constant $0 < \rho_m < 1$ such that

$$\|B(v, u) - B(v, u')\|_m \leq \rho_m \|u - u'\|_m$$

for (v, u) and (v, u') close to $(0, 0)$ in $V \oplus E_m$ where the notion of close depends on the level m. However, we should point out that the stronger contraction assumption (for every contraction constant $0 < \varepsilon < 1$) is crucial later on for the stability of sc^0-contraction germs under perturbations.

Proof of Theorem 2.3. We fix $m \geq 0$ and first show that the set

$$\frac{1}{|b|}\|\delta(v+b) - \delta(v)\|_m \tag{2}$$

is bounded for v and $b \neq 0$ belonging to a small ball around zero in V whose radius depends on m. Since the map B is of class sc^1, there exists at $(v, u) \in U_{m+1}$ a bounded linear map $DB(u, v) \in \mathcal{L}(\mathbb{R}^n \oplus E_m, F_m)$. We introduce the following notation for the partial derivatives,

$$\begin{aligned}
DB(v, u)(\widehat{v}, \widehat{u}) &= DB(v, u)(\widehat{v}, 0) + DB(v, u)(0, \widehat{u}) \\
&= D_1B(v, u)\widehat{v} + D_2B(v, u)\widehat{u}.
\end{aligned}$$

Since $v \mapsto \delta(v)$ is a continuous map into E_{m+1} and since the map B is of class C^1 as a map from an open neighborhood of 0 in $V \oplus E_{m+1}$ into E_m, we have the identity

$$B(v+b, \delta(v+b)) - B(v, \delta(v+b)) = \left(\int_0^1 D_1B(v+sb, \delta(v+b))ds\right) \cdot b.$$

As a consequence,

$$\begin{aligned}
&\frac{1}{|b|}\|B(v+b, \delta(v+b)) - B(v, \delta(v+b))\|_m \\
&\qquad\qquad \leq \int_0^1 \|D_1B(v+sb, \delta(v+b))\|_m ds \leq C_m.
\end{aligned} \tag{3}$$

Recalling $\delta(v) = B(v, \delta(v))$ and $\delta(v+b) = B(v+b, \delta(v+b))$, we have the identity

$$\begin{aligned}
&\delta(v+b) - \delta(v) - (B(v, \delta(v+b)) - B(v, \delta(v))) \\
&\qquad = B(v+b, \delta(v+b)) - B(v, \delta(v+b)).
\end{aligned} \tag{4}$$

From the contraction property of B in the second variable one concludes

$$\|B(v, \delta(v+b)) - B(v, \delta(v))\|_m \leq \rho_m \cdot \|\delta(v+b) - \delta(v)\|_m. \tag{5}$$

Now, using $0 < \rho_m < 1$, one derives from (4) using (3) and (5) the estimate

$$\begin{aligned}
&\tfrac{1}{|b|}\|\delta(v+b)-\delta(v)\|_m \\
&\qquad \le \frac{1}{1-\rho_m}\cdot\frac{1}{|b|}\|B(v+b,\delta(v+b))-B(v,\delta(v+b))\|_m \\
&\qquad \le \frac{1}{1-\rho_m}\cdot C_m
\end{aligned}$$

as claimed in (2). Since B is of class C^1 from $V \oplus E_{m+1}$ into E_m and since $\|\delta(v+b)-\delta(v)\|_{m+1} \le C'_{m+1}\cdot|b|$ by (2), the estimate

$$\delta(v+b)-\delta(v)-DB(v,\delta(v))\cdot(b,\delta(v+b)-\delta(v)) = o_m(b) \tag{6}$$

holds true, where $o_m(b) \in E_m$ is a function satisfying $\frac{1}{|b|}o_m(b) \to 0$ in E_m as $b \to 0$ in V. We next prove

$$\|D_2B(v,\delta(v))h\|_m \le \rho_m\cdot\|h\|_m \tag{7}$$

for all $h \in E_{m+1}$. Fixing $h \in E_{m+1}$, we can estimate

$$\begin{aligned}
&\|D_2B(v,\delta(v))h\|_m \\
&\le \frac{1}{|t|}\cdot\|B(v,\delta(v)+th)-B(v,\delta(v))-D_2B(v,\delta(v))[th]\|_m \\
&\qquad +\frac{1}{|t|}\cdot\|B(v,\delta(v)+th)-B(v,\delta(v))\|_m.
\end{aligned}$$

In view of the postulated contraction property of B, the second term is bounded by $\rho_m\cdot\|h\|_m$, while the first term tends to 0 as $t \to 0$ because B is of class C^1. Hence the claim (7) follows. Using (7) and the fact that E_{m+1} is dense in E_m, we derive for the continuous linear operator $D_2B(v,\delta(v)) : E_m \to E_m$ the bound

$$\|D_2B(v,\delta(v))h\|_m \le \rho_m\cdot\|h\|_m \tag{8}$$

for all $h \in E_m$. Thus, since $\rho_m < 1$, the continuous linear map

$$\begin{gathered}
L(v) : E_m \to E_m \\
L(v) := 1 - D_2B(v,\delta(v))
\end{gathered}$$

is an isomorphism. Applying the inverse $L(v)^{-1}$ to both sides of (6) we obtain the estimate

$$\delta(v+b) - \delta(v) - L(v)^{-1} D_1 B(v, \delta(v)) b = o_m(b)$$

in E_m. Therefore, the map $v \mapsto \delta(v)$ from V into E_m is differentiable and its derivative $\delta'(v) \in \mathcal{L}(\mathbb{R}^n, E_m)$ is given by the formula

$$\delta'(v) = L(v)^{-1} D_1 B(v, \delta(v)). \tag{9}$$

It remains to show that $v \mapsto \delta'(v) \in \mathcal{L}(\mathbb{R}^n, E_m)$ is continuous. To see this we define the map $F : (V \oplus \mathbb{R}^n) \oplus E_m \to E_m$ by setting

$$F(v, b, h) = D_1 B(v, \delta(v)) b + D_2 B(v, \delta(v)) h.$$

The map F is continuous and, in view of (8), it is a contraction in h. Applying a parameter dependent version of Banach's fixed point theorem to F we find a continuous function $(v, b) \mapsto h(v, b)$ from a small neighborhood of 0 in $V \oplus \mathbb{R}^n$ into E_m satisfying $F(v, b, h(v, b)) = h(v, b)$. Since we also have $F(v, b, \delta'(v)b) = \delta'(v)b$, it follows from the uniqueness that $h(v, b) = \delta'(v)b$ and so the map $(v, b) \mapsto \delta'(v)b$ is continuous. Now using the fact that V is contained in a finite dimensional space, we conclude that $v \mapsto \delta'(v) \in \mathcal{L}(\mathbb{R}^n, E_m)$ is a continuous map. The proof of Theorem 2.3 is complete. ■

2.3 Higher Regularity

Theorem 2.3 shows that the sc^0-contraction germ f which is also of class sc^1 has a solution germ δ satisfying $f(v, \delta(v)) = 0$ which is also of class sc^1. Our next aim is to show that if f is of class sc^k, then δ is also of class sc^k. To do so we begin with a construction.

Lemma 2.4. *Assume that $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ is an sc^0-contraction germ and of class sc^k with $k \geq 1$. Denote by δ the solution germ and assume it is of class sc^j. (By Theorem 2.3, δ is at least of class sc^1.) Define the germ $f^{(1)}$ by*

$$f^{(1)} : \mathcal{O}(TV \oplus TE, 0) \to TE$$

$$\begin{aligned} f^{(1)}(v, b, u, w) &= (u - B(v, u), w - DB(v, \delta(v))\,(b, w)) \\ &= (u, w) - B^{(1)}(v, b, u, w), \end{aligned} \tag{10}$$

where the last line defines the map $B^{(1)}$. Then $f^{(1)}$ is an sc^0-contraction germ and of class $sc^{\min\{k-1, j\}}$.

Proof. For v small, the map $B^{(1)}$ has the contraction property with respect to (u, w). Indeed, on the m-level of $(TE)_m = E_{m+1} \oplus E_m$, i.e., for $(u, w) \in E_{m+1} \oplus E_m$ and for v small we can estimate, using (7),

$$\begin{aligned}
&\|B^{(1)}(v, b, u', w') - B^{(1)}(v, b, u, w)\|_m \\
&\quad = \|B(v, u') - B(v, u)\|_{m+1} \\
&\qquad + \|DB(v, \delta(v))(b, w') - DB(v, \delta(v))(b, w)\|_m \\
&\quad \leq \rho_{m+1} \|u' - u\|_{m+1} + \|D_2 B(v, \delta(v))[w' - w]\|_m \\
&\quad \leq \max\{\rho_{m+1}, \rho_m\} \cdot \big(\|u' - u\|_{m+1} + \|w' - w\|_m\big) \\
&\quad = \max\{\rho_{m+1}, \rho_m\} \cdot \|(u', w') - (u, w)\|_m.
\end{aligned}$$

Consequently, the germ $f^{(1)}$ is an sc^0-contraction germ. If now f is of class sc^k and δ of class sc^j, then the germ $f^{(1)}$ is of class $\text{sc}^{\min\{k-1,j\}}$, as one verifies by comparing the tangent map Tf with the map $f^{(1)}$ and using the fact that the solution δ is of class sc^j. By Theorem 2.2, the solution germ $\delta^{(1)}$ of $f^{(1)}$ is at least of class sc^0. It solves the equation

$$f^{(1)}(v, b, \delta^{(1)}(v, b)) = 0. \tag{11}$$

But also the tangent germ $T\delta$ defined by $T\delta(v, b) = (\delta(v), D\delta(v)b)$ is a solution of (11). From the uniqueness we conclude $\delta^{(1)} = T\delta$. ■

For our higher regularity theorem we need the following lemma.

Lemma 2.5. *Assume we are given an sc^0-contraction germ f of class sc^k and a solution germ δ of class sc^j with $j \leq k$. Then there exists an sc^0-contraction germ $f^{(j)}$ of class $\text{sc}^{\min\{k-j,1\}}$ having $\delta^{(j)} := T^j\delta$ as the solution germ.*

Proof. We prove the lemma by induction with respect to j. If $j = 0$ and f is an sc^0-contraction germ of class sc^k, $k \geq 0$, then we set $f^{(0)} = f$ and $\delta^{(0)} = \delta$. Hence the result holds true if $j = 0$. Assuming the result has been proved for j, we show it is true for $j + 1$. Since $j + 1 \geq 1$ and $k \geq j + 1$, the map $f^{(1)}$ defined by (10) is of class $\text{sc}^{\min\{k-1,j+1\}}$ in view of Lemma 2.4. Moreover, the solution germ $\delta^{(1)} = T\delta$ satisfies

$$f^{(1)} \circ \text{gr}(\delta^{(1)}) = 0,$$

and is of class sc^j. Since $\min\{k-1, j+1\} \geq j$, by the induction hypothesis there exists a map $(f^{(1)})^{(j)} =: f^{(j+1)}$ of regularity class $\min\{\min\{k-1, j+1\} - j, 1\} = \min\{k-(j+1), 1\}$ so that

$$f^{(j+1)} \circ \mathrm{gr}((\delta^{(1)})^{(j)}) = 0.$$

Setting $\delta^{(j+1)} = (\delta^{(1)})^{(j)} = T^j(T\delta) = T^{j+1}\delta$ the result follows. ■

Now comes the main result of this section.

Theorem 2.6 (Germ-Implicit Function Theorem). *If $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ is an sc^0-contraction germ which is, in addition, of class sc^k, then the solution germ*

$$\delta : \mathcal{O}(V, 0) \to (E, 0)$$

satisfying

$$f(v, \delta(v)) = 0$$

is also of class sc^k.

In case the germ f is sc-smooth at 0, it follows that for every m and k there is an open neighborhood $V_{m,k}$ of 0 in V on which the map $v \mapsto \delta(v)$ goes into the m-level and belongs to C^k. In particular, the solution δ is sc-smooth at the smooth point 0. The above theorem will be one of the key building blocks for all future versions of implicit function theorems as well as for the transversality theory.

Proof. Arguing by contradiction assume that the solution germ δ is of class sc^j but not of class sc^{j+1} with $j < k$. In view of Lemma 2.5, there exists an sc^0-contraction germ $f^{(j)}$ of class $\mathrm{sc}^{\min\{k-j,1\}}$ so that $\delta^{(j)} = T^j\delta$ satisfies

$$f^{(j)} \circ \mathrm{gr}(\delta^{(j)}) = 0.$$

Since also $k - j \geq 1$, it follows that $f^{(j)}$ is at least of class sc^1. Consequently, the solution germ $\delta^{(j)}$ is at least of class sc^1. Since $\delta^{(j)} = T^j\delta$, we conclude that δ is at least of class sc^{j+1} contradicting our assumption. The proof of the theorem is complete. ■

The same discussion remains valid for germs f defined on $V \oplus E$, where V is any finite-dimensional partial quadrant.

Theorem 2.7. *Let V be a finite-dimensional partial quadrant. If $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ is an sc^0-contraction germ which is, in addition, of class sc^k, then the solution germ*

$$\delta : \mathcal{O}(V, 0) \to (E, 0)$$

satisfying

$$f(v, \delta(v)) = 0$$

is also of class sc^k.

3 Fredholm Sections

In this section we introduce the notion of a Fredholm section and develop its local theory.

3.1 Regularizing Sections

Let $p : Y \to X$ be a strong M-polyfold bundle as defined in Definition 4.9 in [12]. By $\Gamma(p)$ we denote the vector space of sc-smooth sections of the bundle p and by $\Gamma^+(p)$ the linear subspace of sc$^+$-sections. The following definition can be viewed as a formalization of "elliptic bootstrapping".

Definition 3.1. *A section $f \in \Gamma(p)$ is said to be* **regularizing** *provided $f(q) \in Y_{m,m+1}$ for a point $q \in X$ implies that $q \in X_{m+1}$. We denote by $\Gamma_{reg}(p)$ the subset of $\Gamma(p)$ consisting of regularizing sections.*

We observe that if $f \in \Gamma_{reg}(p)$ and $s \in \Gamma^+(p)$, then the sum $f + s$ belongs to $\Gamma_{reg}(p)$. Indeed, if $(f + s)(q) = y \in Y_{m,m+1}$, then $q \in X_m$. Hence $f(q) = y - s(q) \in Y_{m,m+1}$ implies $q \in X_{m+1}$ since f is regularizing. Therefore, we have the map

$$\Gamma_{reg}(p) \times \Gamma^+(p) \to \Gamma_{reg}(p)$$

defined by $(f, s) \to f + s$. Note however that the sum of two regularizing sections is, in general, not regularizing.

3.2 Fillers and Filled Versions

The filler is a very convenient technical devise in the local investigation of the solution set of a section of a strong M-polyfold bundle. Namely, it turns the local study of the section, which perhaps is defined on a complicated space

having varying dimensions into the equivalent local study of a filled section which is defined on a fixed open set of a partial quadrant in an sc-Banach space and which has its image in a fixed sc-Banach space. In the following we summarize the discussion about fillers from [12]. We consider the fillable strong M-polyfold bundle $p : Y \to X$ as defined in Definition 4.10 in [12]. Then, given the section f of the bundle p and a smooth point $q \in X$, there exists locally a filled version of f which is an sc-smooth map

$$\bar{\mathbf{f}} : \widehat{O} \to F$$

between an open set $\widehat{O}$ of a partial cone in an sc-Banach space and an sc-Banach space F. This filled version is obtained as follows.

We choose a strong bundle chart (as in Definition 4.8 in [12])

$$\Phi : Y|U \to K^{\mathcal{R}}|O$$

covering the sc-diffeomorphism $\varphi : U \to O$ between an open neighborhood $U \subset X$ of the point q and the open subset O of the splicing core $K^{\mathcal{S}} = \{(v, e) \in V \oplus E|\, \pi_v(e) = e\}$ associated with the splicing $\mathcal{S} = (\pi, E, V)$. Here V is an open subset of a partial quadrant in an sc-Banach space W and E is an sc-Banach space. We assume that $\varphi(q) = 0$.

The bundle $K^{\mathcal{R}}|O$ is defined by $K^{\mathcal{R}}|O := \{(w, u) \in O \oplus F|\, \rho_w(u) = u\}$ where we have abbreviated $w = (v, e)$. The bundle is the splicing core of the strong bundle splicing $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ in which F is an sc-Banach space.

In the notation of [12] which is recalled in the glossary

$$K^{\mathcal{R}} = K^{\mathcal{R}^0} = K^{\mathcal{R}}(0).$$

Having the splicing $\mathcal{S} = (\pi, E, V)$, we introduce the complementary splicing $\mathcal{S}^c = (1-\pi, E, O)$ whose associated splicing core is given by $K^{\mathcal{S}^c} = \{(v, \varepsilon) \in V \oplus E|\, \pi_v(\varepsilon) = 0\}$. According to the decomposition $E = \pi_v(E) \oplus (1 - \pi_v)(E)$ every element of E has the unique representation

$$e = e' + \varepsilon, \quad \pi_v(e') = e', \quad \pi_v(\varepsilon) = 0.$$

Using the natural projections $O \to V$ and $K^{\mathcal{S}^c} \to V$ we can form the Whitney-sum $O \oplus_V K^{\mathcal{S}^c} \subset K^{\mathcal{S}} \oplus_V K^{\mathcal{S}^c}$ which can be identified with the open subset of $\widehat{O} \subset V \oplus E$ defined by

$$\begin{aligned}\widehat{O} &= \{(v, e) \in V \oplus E|\, (v, \pi_v(e)) \in O\}\\ &= \{(v, e' + \varepsilon) \in V \oplus E|\, \pi_v(e') = e',\, \pi_v(\varepsilon) = 0, \text{ and } (v, e') \in O\}.\end{aligned}$$

We define the projection $r:\widehat{O}\to O$ by $r(v,e)=(v,\pi_v(e))$.Then the preimage of a point $(v,\pi_v(e))$,

$$r^{-1}(v,\pi_v(e))=\{(v,\pi_v(e))\}\times\ker\pi_v\subset\{(v,\pi_v(e))\}\times E,$$

has the structure of a Banach space so that we may view $r:\widehat{O}\to O$ set-theoretically as a bundle over O.

Associated with the strong bundle splicing $\mathcal{R}=(\rho,F,(O,\mathcal{S}))$ there is the complementary strong bundle splicing $\mathcal{R}^c=(1-\rho,F,(O,\mathcal{S}))$ whose splicing core is given by

$$K^{\mathcal{R}^c}=\{(w,u)\in O\oplus F|\,\rho_w(u)=0\}$$

where we have abbreviated $w=(v,e)\in O$.

By definition, a **filler** for $\mathcal{R}$ is an sc-smooth map

$$f^c:\widehat{O}\to K^{\mathcal{R}^c}$$

covering the identity $O\to O$ so that f^c is fiber-wise a **linear isomorphism**. More in detail, for every $(v,e')\in O$, the map

$$\begin{aligned}\ker\pi_v&\to\ker\rho_{(v,e')}\\ \varepsilon&\mapsto f^c(v,e'+\varepsilon)\end{aligned}$$

is a linear isomorphism.

If f is a given section of $K^{\mathcal{R}}\to O$, we introduce the composition

$$f\circ r:\widehat{O}\to K^{\mathcal{R}}$$

and define the **filled section** $\overline{f}$ of the bundle $\widehat{O}\triangleleft F\to\widehat{O}$ by the formula

$$\begin{aligned}\overline{f}(v,e)&=((v,e),\overline{\mathbf{f}}(v,e))\\ &=f\circ r(v,e)+f^c(v,e)\end{aligned}$$

for all $(v,e)\in\widehat{O}$. The principal part $\overline{\mathbf{f}}:\widehat{O}\to F$ of the filled section $\overline{f}$ is an sc-smooth map which splits into a sum of two sc-smooth maps

$$\overline{\mathbf{f}}(v,e)=\mathbf{f}(v,e)+\mathbf{f}^c(v,e)\in\ker(1-\rho_{(v,\pi_v(e))})\oplus\ker\rho_{(v,\pi_v(e))}=F$$

where $\mathbf{f}$ is the principal part of the given section $f\circ r:\widehat{O}\to K^{\mathcal{R}}$ and $\mathbf{f}^c$ is the principal part of the filler $f^c:\widehat{O}\to K^{\mathcal{R}^c}$.

Now we assume that $(v, e) \in \widehat{O}$ is a zero of the filled section,

$$\overline{\mathbf{f}}(v, u) = 0.$$

Then $\mathbf{f}(v, e) = 0$ and $\mathbf{f}^c(v, e) = 0$. We set $e = e' + \varepsilon$ with $\pi_v(e') = e'$ and $\pi_v(\varepsilon) = 0$. By assumptions of the filler f^c, the map $\varepsilon \mapsto \mathbf{f}^c(v, e' + \varepsilon)$ is a linear isomorphism. We conclude that $\varepsilon = 0$ and so $r(v, e) = (v, \pi_v(e)) = (v, e) \in O$. Therefore,

$$0 = \mathbf{f}(v, e),$$

implying that (v, e) is a zero of the given section f of the bundle $K^{\mathcal{R}} \to O$.

Having the above construction in mind we recall from [12] that a strong M-polyfold bundle $p : Y \to X$ is called **fillable** if around every point $q \in X$ there exists a compatible strong M-polyfold bundle chart $(U, \Phi, (K^{\mathcal{R}}, \mathcal{R}))$ whose strong bundle splicing $\mathcal{R}$ admits a filler as introduced above. In principle there can be many different bundle charts admitting different fillers.

The representation of a given section g of the M-polyfold bundle $p : Y \to X$ in the above strong bundle chart $\Phi : Y|O \to K^{\mathcal{R}}|O$ is the push-forward

$$f := \Phi_* g = \Phi \circ g \circ \varphi^{-1}$$

which is an sc-smooth section of the above strong bundle chart $K^{\mathcal{R}} \to O$. Following the above construction we consider the filled section $\overline{f}$ of the local bundle $\widehat{O} \triangleleft F \to \widehat{O}$ given by

$$\overline{f}(v, e) = f \circ r(v, e) + f^c(v, e).$$

Then $\overline{f}(v, e) = 0$ if and only if $r(v, e) = (v, e)$, i.e., $(v, e) \in O$, so that $x = \varphi^{-1}(v, e)$ solves $g(x) = 0$.

Now we assume that the smooth point $q \in X$ corresponds to $\varphi(q) = (0, 0) \in O \subset \widehat{O}$ and assume also that

$$f(0, 0) = 0.$$

Then the linearization of the local section f at the solution $(0, 0)$,

$$f'(0, 0) : T_{(0,0)}O \to \ker(1 - \rho_{(0,0)}),$$

is equal to $D\overline{\mathbf{f}}|T_{(0,0)}O$ where the tangent space is given by

$$T_{(0,0)}O = \{(\delta w, \delta e) \in W \oplus E|\, \pi_0(\delta e) = \delta e\}.$$

In [12] we have verified the following relation between $f'(0,0)$ and the linearization

$$D\bar{\mathbf{f}}(0,0): W \oplus E \to F$$

of the filled section $\bar{\mathbf{f}}$. Abbreviating the splittings $E = \ker(1-\pi_0) \oplus \ker \pi_0 = E^+ \oplus E^-$ and $F = \ker(1-\rho_{(0,0)}) \oplus \ker \rho_{(0,0)} = F^+ \oplus F^-$, we use the notation $\delta e = (\delta a, \delta b) \in E^+ \oplus E^-$. Then

$$D\bar{\mathbf{f}}(0,0): (W \oplus E^+) \oplus E^- \to F^+ \oplus F^-$$

has the following matrix form

$$\begin{bmatrix} \delta w \oplus \delta a \\ \delta b \end{bmatrix} \mapsto \begin{bmatrix} f'(0,0) & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \delta w \oplus \delta a \\ \delta b \end{bmatrix}.$$

where

$$C = \mathbf{f}^c(0,\cdot): E^- \to F^-$$

is an sc-isomorphism by the definition of a filler.

We conclude that the linearization $f'(0,0)$ of the section $f = \Phi_* g$ is surjective if and only if the linearization $D\bar{\mathbf{f}}(0,0)$ of the filled section is surjective. Moreover, the kernel $\ker f'(0,0)$ is equal to the kernel of $D\bar{\mathbf{f}}(0,0)$ in the sense that

$$\ker f'(0,0) \oplus \{0\} = \ker D\mathbf{f}^c(0,0)$$

where the splitting on the left side refers to the $(W \oplus E^+) \oplus E^-$. One also reads off from the matrix form that $f'(0,0)$ is an sc-Fredholm operator if and only if $D\bar{\mathbf{f}}(0,0)$ is an sc-Fredholm operator and in this case their Fredholm indices agree.

Summarizing, the properties of a section germ $[g,q]$ of a fillable strong M-polyfold bundle $p: Y \to X$ are fully reflected in the properties of the filled section germ $[\bar{f},(0,0)]$ of the local bundle $\widehat{O} \triangleleft F \to \widehat{O}$.

Let us finally point out once more that the fillers are not unique. A section germ $[f,q]$ of a fillable M-polyfold bundle has, in general, many filled versions. They all have the same properties.

3.3 Basic sc-Germs and Fredholm Sections

The aim of this section is the introduction of the crucial concept of a Fredholm section of a fillable strong M-polyfold bundle. We start with local considerations and consider sc-smooth germs

$$f: \mathcal{O}(C,0) \to F,$$

where $C \subset E$ is a partial quadrant in the sc-Banach space E and where F is another sc-Banach space. We do not require that $f(0) = 0$ but we observe that $f(0)$ is also a smooth point, since 0 is a smooth point and f is an sc-smooth map. We view these germs as sc-smooth section germs of the local strong bundle $C \triangleleft F \to C$. The definition of a special local strong bundle from [12] is recalled in the glossary.

The product $C \triangleleft F$ is the product $C \oplus F$ equipped with the two filtrations

$$(C \triangleleft F)_{m,m} = C_m \oplus F_m \quad \text{and} \quad (C \triangleleft F)_{m,m+1} = C_m \oplus F_{m+1}$$

for all $m \geq 0$. A map $\Phi : C \triangleleft F \to C' \triangleleft F'$ which is of the form

$$\Phi(x, h) = (a(x), b(x, h))$$

and linear in h is called a strong bundle isomorphism if Φ is a bijection and Φ and its inverse are of class $\text{sc}^\infty_\triangleleft$. The notion of an $\text{sc}^\infty_\triangleleft$-map introduced in [12] is recalled in the glossary.

Definition 3.2. *Two sc-smooth germs $f : \mathcal{O}(C, 0) \to F$ and $g : \mathcal{O}(C', 0) \to F'$ are called* **strongly equivalent***, denoted by $f \sim g$, if there exists a germ of strong bundle isomorphism $\Phi : C \triangleleft F \to C' \triangleleft F'$ near 0 covering the sc-diffeomorphism germ $\varphi : (C, 0) \to (C', 0)$ so that g equals the push-forward $\Phi_* f = \Phi \circ f \circ \varphi^{-1}$.*

We recall from Section 4.4 in [12] that an sc-smooth germ $[f, 0]$ is called linearized Fredholm at the smooth point 0, if the linearization

$$f'_{[s]}(0) := D(f - s)(0) : E \to F$$

is an sc-Fredholm operator, where $[s, 0]$ is any sc^+-germ satisfying $s(0) = f(0)$. The integer

$$\text{Ind}(f, 0) := i(f'_{[s]}(0)),$$

where i on the right hand side denotes the Fredholm index, is independent of the sc^+-germ $[s, 0]$ as is demonstrated in [12]. Using the chain rule for sc-smooth maps, the following proposition follows immediately from the definitions.

We denote by $\mathfrak{C}$ the class of all section germs.

Proposition 3.3. *If $[f, 0]$ and $[g, 0]$ are strongly equivalent elements in $\mathfrak{C}$ and one of them is linearized Fredholm at 0, then so is the other and in this case their Fredholm indices agree,*

$$\text{Ind}(f, 0) = \text{Ind}(g, 0).$$

Among all elements of the section germs in $\mathfrak{C}$ there is a distinguished class of special elements called basic class, denoted by $\mathfrak{C}_{basic}$ and defined as follows.

Definition 3.4. *The* **basic class** $\mathfrak{C}_{basic}$ *consists of all sc-smooth germs* $[g, 0]$ *in* $\mathfrak{C}$ *of the form*

$$g : \mathcal{O}(([0, \infty)^k \oplus \mathbb{R}^{n-k}) \oplus W, 0) \to \mathbb{R}^N \oplus W,$$

where W *is an sc-Banach space, so that with the projection* $P : \mathbb{R}^N \oplus W \to W$*, the germ*

$$P \circ (g - g(0, 0)) : \mathcal{O}(([0, \infty)^k \oplus \mathbb{R}^{n-k}) \oplus W, 0) \to W$$

is an sc^0*-contraction germ in the sense of Definition 2.1.*

The basic elements have special properties as the following proposition demonstrates.

Proposition 3.5. *A germ* $g : \mathcal{O}(([0, \infty)^k \oplus \mathbb{R}^{n-k}) \oplus W, 0) \to \mathbb{R}^N \oplus W$ *in* $\mathfrak{C}_{basic}$ *is linearized Fredholm at* 0 *and its Fredholm index is equal to*

$$\operatorname{Ind}(g, 0) = n - N.$$

Proof. Abbreviate the partial quadrant $[0, \infty)^k \oplus \mathbb{R}^{n-k}$ in $\mathbb{R}^n$ by C. Denoting by (r, w) the elements in $C \oplus W$, we take the sc-germ

$$s : \mathcal{O}(C \oplus W, 0) \to \mathbb{R}^N \oplus W$$

defined as the constant section $s(r, w) = g(0, 0)$. Since $g(0, 0)$ is a smooth point, the section s is an sc^+-section. The germ $f = g - s = g - g(0, 0)$ satisfies $f(0) = 0$. By assumption, the map

$$P \circ \big(g(r, w) - g(0, 0)\big) = w - B(r, w)$$

is an sc^0-contraction germ in the sense of Definition 2.1. Hence the linearization

$$Df(0, 0) : \mathbb{R}^n \oplus W \to \mathbb{R}^N \oplus W$$

of the section f at the point $(0, 0) \in C \oplus W$ is given by the formula

$$
\begin{aligned}
&Df(0,0)(\delta r, \delta w)\\
&\quad = \delta w - D_2B(0,0)\delta w\\
&\qquad - D_1B(0,0)\delta r + (I-P)Dg(0,0)(\delta r, \delta w).
\end{aligned}
$$

As in the proof of Theorem 2.3 one verifies that the linear map $\delta w \mapsto D_2B(0,0)\delta w$ from W_m to W_m is a contraction for every $m \geq 0$, so that

$$
I - D_2B(0,0) : W \to W
$$

is an sc-isomorphism. Therefore, the sc-operator

$$
\begin{aligned}
\mathbb{R}^n \oplus W &\to \mathbb{R}^N \oplus W\\
(\delta r, \delta w) &\mapsto (0, \delta w - D_2B(0,0)\delta w)
\end{aligned}
\tag{12}
$$

is an sc-Fredholm operator whose kernel is equal to $\mathbb{R}^n \oplus \{0\}$ and whose cokernel is equal to $\mathbb{R}^N \oplus \{0\}$, so that the Fredholm index is equal to $n - N$. We shall show that the linearized sc-operator is an sc$^+$-perturbation of the operator (12). Indeed, since $r \mapsto Pg(r, 0)$ is an sc$^+$-section, its linearization $\delta r \mapsto D_1B(0,0)\delta r$ is an sc$^+$-operator. Moreover, because $I - P$ has its image in a finite-dimensional smooth subspace, the operator $(\delta r, \delta w) \to (I - P)Dg(0,0)(\delta r, \delta w)$ is also an sc$^+$-operator. By Proposition 2.11 in [12], the perturbation of an sc-Fredholm operator by an sc$^+$-operator is again an sc-Fredholm operator. Therefore, we conclude that the operator $Df(0,0) : \mathbb{R}^n \oplus W \to \mathbb{R}^N \oplus W$ is an sc-Fredholm operator. Because an sc$^+$-operator is compact if considered on the same level, the Fredholm index is unchanged and $\mathrm{Ind}(g, 0) = n - N$ as claimed. ■

Finally, we are in the position to introduce the crucial concept of a polyfold Fredholm section.

Definition 3.6. *Let $p : Y \to X$ be a fillable strong M-polyfold bundle. An sc-smooth section f of the bundle p is called a* **(polyfold) Fredholm section** *if it possesses the following two properties*

- (**Regularization property**) *The section f is regularizing according to Definition 3.1.*
- (**Basic germ property**) *For every smooth $q \in X$, there* **exists a filled version $\bar{\mathbf{f}} : \widehat{O} \to F$** *whose germ $[\bar{\mathbf{f}}, 0]$ is strongly equivalent to an element in $\mathfrak{C}_{basic}$.*

In order to keep the notation short, we shall skip the word "polyfold" and refer to these sections simply as to Fredholm sections.

As we shall prove next, a linearization $f'_{[s]}(q) : T_qX \to Y_q$ of a Fredholm section at a smooth point $q \in X$, is an sc-Fredholm operator.

Definition 3.7. *If $q \in X$ is a smooth point, we call a section germ $[f, q]$ of a fillable strong M-polyfold bundle a* **Fredholm germ**, *if f is regularizing locally near q and a filled version $[\bar{\mathbf{f}}, 0]$ is strongly equivalent to an element in $\mathfrak{C}_{basic}$.*

Proposition 3.8. *Let $[f, q]$ be a Fredholm germ of the fillable strong M-polyfold bundle $p : Y \to X$ at the smooth point q. Assume a filled version*

$$\bar{\mathbf{f}} : \mathcal{O}(\widehat{O}, 0) \to F$$

is equivalent to the basic element g in $\mathfrak{C}_{basic}$ given by

$$g : \mathcal{O}(([0, \infty)^k \oplus \mathbb{R}^{n-k}) \oplus W, 0) \to \mathbb{R}^N \oplus W.$$

Then any linearization $f'_{[s]}(q) : T_qX \to Y_q$ is an sc-Fredholm operator and the Fredholm index $\operatorname{Ind}(f, q) := i(f'_{[s]}(q))$ satisfies

$$\operatorname{Ind}(f, q) = n - N.$$

Proof. From Proposition 4.16 in [12] we know that a linearization $f'_{[s]}(q)$ is an sc-Fredholm operator if and only if the linearization $\bar{\mathbf{f}}'_{[t]}(0)$ of a filled version is an sc-Fredholm operator. The equivalence relation $\sim$ in Definition 3.2 respects the Fredholm property as well as the Fredholm index. Hence the result follows from Proposition 3.5. ■

3.4 Stability of Fredholm sections

Let us assume that $p : Y \to X$ is a fillable strong M-polyfold bundle. Then $p^1 : Y^1 \to X^1$ is also a fillable strong M-polyfold bundle. The filtrations are defined by $(X^1)_m = X_{m+1}$ and $(Y^1)_{m,k} = Y_{m+1,k+1}$ for $m \geq 0$ and $0 \leq k \leq m + 1$. We denote by $\mathcal{F}(p)$ the collections of all Fredholm sections of the bundle p and by $\Gamma^+(p)$ the vector space of all sc$^+$-sections. The main result in this subsection is the following stability theorem for Fredholm sections.

Theorem 3.9. *Let $p : Y \to X$ be a fillable strong M-polyfold bundle. If $f \in \mathcal{F}(p)$ is a Fredholm section of p and $s \in \Gamma^+(p)$ an sc$^+$-section, then $f+s$ is a Fredholm section of the bundle $p^1 : Y^1 \to X^1$, i.e., $f+s \in \mathcal{F}(p^1)$, and the Fredholm indices satisfy*

$$\mathrm{Ind}(f, q) = \mathrm{Ind}(f+s, q)$$

for every smooth point $q \in X$. Further, if $s_1, ..., s_k \in \Gamma^+(p)$, then the map

$$F : \mathbb{R}^k \oplus X \to Y, \quad F(\lambda, x) = f(x) + \sum_{j=1}^{k} \lambda_j s_j(x)$$

is a Fredholm section of the bundle $(r^*Y)^1 \to (\mathbb{R}^k \oplus X)^1$, where the map $r : \mathbb{R}^k \oplus X \to X$ is the projection and r^*Y the pull-back bundle. Moreover,

$$\mathrm{Ind}(F, (0, q)) = \mathrm{Ind}(f, q) + k.$$

Proof. In view of $s \in \Gamma^+(p)$, with f also the section $f+s$ has the regularizing property. Since under the push-forwards of strong bundle maps the sc$^+$-sections stay sc$^+$-sections, it is sufficient to prove the result in a very special situation. Namely, we may assume that f is already filled and has the contraction normal form. More precisely, we assume that the section f is of the form

$$f : O \subset ([0, \infty)^k \oplus \mathbb{R}^{n-k}) \oplus W \to \mathbb{R}^N \oplus W,$$

where O is a relatively open neighborhood of $(0, 0)$ which corresponds to q. With the projection $P : \mathbb{R}^N \oplus W \to W$, the expression

$$P[f(v, w) - f(0, 0)] = w - B(v, w)$$

has the contraction germ property near $(0, 0)$.

The linearization of the section s with respect to the variable $w \in W$ at the point 0 is denoted by $D_2 s(0)$. Since s is an sc$^+$-section and since the spaces $W_{m+1} \subset W_m$ are compactly embedded, the linear operator $D_2 s(0) : W_m \to \mathbb{R}^N \oplus W_m$ is a compact operator for every level $m \geq 0$. Therefore, introducing $A := P D_2 s(0) : W \to W$ and using Proposition 2.11 in [12], the operator $1 + A : W \to W$ is an sc- Fredholm operator (as defined in Definition 2.8 in [12]) of index 0. The associated sc-decompositions of W are as follows,

$$1 + A : W = C \oplus X \to W = R \oplus Z,$$

where $C = \ker(1 + A)$ and $R = \text{range } (1 + A)$ and $\dim C = \dim Z < \infty$. Because the section s is of class C^1 on every level $m \geq 1$, one deduces the representation

$$P[s(a, w) - s(0, 0)] = Aw + S(a, w)$$

where $D_2 S(0, 0) = 0$. Hence S is a contraction (with arbitrarily small contraction constant) with respect to the second variable on every level $m \geq 1$ if a and w are sufficiently small depending on the level m and the contraction constant. We can write

$$\begin{aligned} P[(f + s)(a, w) - (f + s)(0, 0)] &= w - B(a, w) + Aw + S(a, w) \\ &= (1 + A)w - [B(a, w) - S(a, w)] \\ &= (1 + A)w - \overline{B}(a, w), \end{aligned}$$

where we have abbreviated

$$\overline{B}(a, w) = B(a, w) - S(a, w).$$

By assumption, the map B belongs to the sc^0-contraction germ and hence the map $\overline{B}$ is a contraction in the second variable on every level $m \geq 1$ with arbitrary small contraction constant $\varepsilon > 0$ if a and w are sufficiently small depending on the level m and the contraction constant ε. Denoting the canonical projections by

$$\begin{aligned} P_1 &: W = C \oplus X \to X \\ P_2 &: W = R \oplus Z \to R, \end{aligned}$$

we abbreviate the map

$$\begin{aligned} \varphi(a, w) &:= P_2 \circ P \circ [(f + s)(a, w) - (f + s)(0, 0)] \\ &= P_2[(1 + A)w - \overline{B}(a, w)] \\ &= P_2[(1 + A)P_1 w - \overline{B}(a, w)]. \end{aligned}$$

We have used the relation $(1+A)(1-P_1) = 0$. The operator $L := (1+A)|X : X \to R$ is an sc-isomorphism. In view of $L^{-1} \circ P_2 \circ (1 + A)P_1 w = P_1 w$, we obtain the formula

$$L^{-1} \circ \varphi(a, w) = P_1 w - L^{-1} \circ P_2 \circ \overline{B}(a, w).$$

Writing $w = (1 - P_1)w \oplus P_1 w$, we shall consider $(a, (1 - P_1)w)$ as our new parameter and correspondingly define the map $\widehat{B}$ by

$$\widehat{B}((a, (1 - P_1)w), P_1 w) = L^{-1} \circ P_2 \circ \overline{B}(a, (1 - P_1)w + P_1 w).$$

Since $\overline{B}(a, w)$ is a contraction in the second variable on every level $m \geq 1$ with arbitrary small contraction constant if a and w are sufficiently small depending on the level m and the contraction constant, the right hand side of

$$L^{-1} \circ \varphi(a, (1 - P_1)w + P_1 w) = P_1 w - \widehat{B}(a, (1 - P_1)w, P_1 w)$$

possesses the required contraction normal form with respect to the variable $P_1 w$ on all levels $m \geq 1$, again if a and w are small enough depending on m and the contraction constant.

It remains to prove that the above normal form is the result of an admissible coordinate transformation of the perturbed section $f + s$. Choose a linear isomorphism $\tau : Z \to C$ and define the fiber transformation $\Psi : \mathbb{R}^N \oplus W \to \mathbb{R}^N \oplus X \oplus C$ by

$$\Psi(\delta a \oplus \delta w) := \delta a \oplus L^{-1} \circ P_2 \cdot \delta w \oplus \tau \circ (1 - P_2) \cdot \delta w.$$

We shall view Ψ as a strong bundle map covering the sc-diffeomorphism $\psi : V \oplus W \to V \oplus C \oplus X$ defined by $\psi(a, w) = (a, (1 - P_1)w, P_1 w)$ where $V = [0, \infty)^k \oplus \mathbb{R}^{n-k}$. With the canonical projection

$$\begin{aligned} \overline{P} &: (\mathbb{R}^N \oplus C) \oplus X \to X \\ \overline{P}&(a \oplus (1 - P_1)w \oplus P_1 w) = P_1 w, \end{aligned}$$

and the relation $\overline{P} \circ \Psi \circ (1 - P) = 0$, we obtain the desired formula

$$\begin{aligned} \overline{P} \circ \Psi[(f + s) \circ \psi^{-1}&(a, (1 - P_1)w, P_1 w) - (f + s) \circ \psi^{-1}(0, 0, 0)] \\ &= P_1 w - \widehat{B}(a, (1 - P_1)w, P_1 w). \end{aligned}$$

We have proved that $[f + s, q]$ is a strong polyfold Fredholm germ of the bundle $[b^1, q]$. Since at 0 the linearisations of f and $f + s$ only differ by an sc^+-operator the Fredholm indices are the same.

The second part of the theorem follows along the same lines and is left to the reader. The proof of Theorem 3.9 is complete. ■

In summary, starting with a fillable strong M-polyfold bundle $p : Y \to X$ and a Fredholm section f of p we can consider the set of perturbations $\{f + s \mid s \in \Gamma^+(p)\}$. These are all Fredholm sections, not of the bundle p but of the bundle of $p^1 : Y^1 \to X^1$. As we shall see later, most of these sections $f + s$ are transversal to the zero section in the sense that at a zero q, which by the regularizing property has to be smooth, the linearization

$$(f + s)'(q) : T_q X^1 \to Y_q^1$$

is a surjective sc-Fredholm operator. We shall also see that the solution set near q admits the structure of a smooth finite-dimensional manifold in a natural way. If the underlying M-polyfold X has a boundary, a generic sc$^+$-perturbation s will not only make the section $f + s$ transversal to the zero-section but it will also put the solution set of $f + s = 0$ into a general position to the boundary ∂X so that the solution set is a smooth manifold with boundary with corners.

4 Local Solutions of Fredholm Sections

In this section we shall show that a Fredholm section f of a fillable strong M-polyfold bundle $p : Y \to X$ near a solution q of $f(q) = 0$ has a smooth solution manifold provided the linearization $f'(q) : T_q X \to Y_q$ is surjective. We first consider the case in which the solution q of $f(q) = 0$ does not belong to the boundary ∂X of the M-polyfold X. The boundary case will be studied in section 4.3. The boundary ∂X of the M-polyfold X is defined by means of the degeneracy index $d : X \to \mathbb{N}$ introduced in Section 3.4 of [12].

For the convenience of the reader we therefore recall first the definition of the degeneracy index and the definition of a face of the M-polyfold X. Around a point $x \in X$ we take a M-polyfold chart $\varphi : U \to K^{\mathcal{S}}$ where $K^{\mathcal{S}}$ is the splicing core associated with the splicing $\mathcal{S} = (\pi, E, V)$. Here V is an open subset of a partial quadrant C contained in the sc-Banach space W. By definition there exists a linear isomorphism from W to $\mathbb{R}^n \oplus Q$ mapping C onto $[0, \infty)^n \oplus Q$. Identifying the partial quadrant C with $[0, \infty)^n \oplus Q$ we shall use the notation $\varphi = (\varphi_1, \varphi_2) \in [0, \infty)^n \oplus (Q \oplus E)$ according to the splitting of the target space of φ. We associate with the point $x \in U$ the integer $d(x)$ defined by

$$d(x) = \sharp\{\text{coordinates of } \varphi_1(x) \text{ which are equal to } 0\}.$$

By Theorem 3.11 in [12], the integer d does not depend on the choice of the M-polyfold chart used. A point $x \in X$ satisfying $d(x) = 0$ is called an interior point of X. The set ∂X of the boundary points of X is defined as

$$\partial X = \{x \in X \mid d(x) > 0\}.$$

A point $x \in X$ satisfying $d(x) = 1$ is called a **good boundary point**. A point satisfying $d(x) \geq 2$ is called a corner and $d(x)$ is the order of this corner.

Definition 4.1. *The closure of a connected component of the set* $X(1) = \{x \in X \mid d(x) = 1\}$ *is called a* **face** *of the M-polyfold* X*.*

Around every point $x_0 \in X$ there exists an open neighborhood $U = U(x_0)$ so that every $x \in U$ belongs to precisely $d(x)$ many faces of U. This is easily verified. Globally it is always true that $x \in X$ belongs to at most $d(x)$ many faces and the strict inequality is possible.

4.1 Good Parameterizations

We assume that $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ is a fillable strong bundle splicing. The splicing $\mathcal{S}$ is the triple $\mathcal{S} = (\pi, E, V)$. We begin with the case in which V **is an open subset of the sc-Banach space** G. This is equivalent to requiring that the open set O in the splicing core $K^{\mathcal{S}} \subset G \oplus E$ has no boundary, i.e., that the degeneracy index vanishes. Associated with the splicing $\mathcal{R}$ is the splicing core $K = K^{\mathcal{R}} = \{((v, e), u) \in O \oplus F \mid \rho_{(v,e)}(u) = u\}$ and the local strong M-polyfold bundle

$$b : K \to O.$$

Now we study the linearized Fredholm section f of the bundle b. Given a smooth point $q \in O$ satisfying $f(q) = 0$, we can linearize the section f at this point and obtain the linear sc-Fredholm operator

$$f'(q) : T_q O \to K_q := b^{-1}(q).$$

Its kernel $N = \ker f'(q)$ is finite dimensional and consists of smooth points in view of Proposition 2.9 in [12]. Hence according to Proposition 2.7 in [12], the kernel N possesses an sc-complement. Any sc-complement of N in $G \oplus E$ will be denoted by $N^{\perp}$, so that

$$N \oplus N^{\perp} = G \oplus E.$$

The following definition of a good parametrization will be useful in the description of the zero set $\{f = 0\}$ of a Fredholm section f of a fillable M-polyfold bundle $b : K \to O$ near a zero q at which the linearization $f'(q)$ is surjective.

In view of the traditional approach to Fredholm theory, the definition might look at first sight like what ought to be the consequences of the assumption that the regularizing section has at every zero q a linearization $f'(q)$ which is a surjective Fredholm operator. However, as we shall see later on (Proposition 4.6) the existence of a good parametrization requires much stronger hypotheses than simply linearized Fredholm and $f'(q)$ surjective, because in our setting of M-polyfolds we do not have the familiar implicit function theorem available in order to deduce the behavior of f near a point q from the properties of its linearization $f'(q)$ at the point q. This is the reason for the introduction of the new concept of a Fredholm section in Definition 3.6.

Definition 4.2. *Consider a local strong bundle $b : K \to O$ where O has no boundary. Assume that the section f of the bundle b is regularizing and at every solution $p \in O$ of $f(p) = 0$ the section f is linearized Fredholm and its linearization $f'(p) : T_pO \to K_p$ is surjective. If $q \in O$ is a solution of $f(q) = 0$, the section f is said to have a* **good parametrization of its solution set near q**, *if there exist an open neighborhood $Q \subset N$ of 0 in the kernel N of $f'(q)$, an open neighborhood $U(q)$ in O, and an sc-smooth map*

$$A : Q \to N^{\perp}$$

into a complement $N^{\perp}$ such that the following holds true.

(1) $A(0) = 0$ and $DA(0) = 0$.

(2) The map

$$\Gamma : Q \to G \oplus E, \quad n \mapsto q + n + A(n)$$

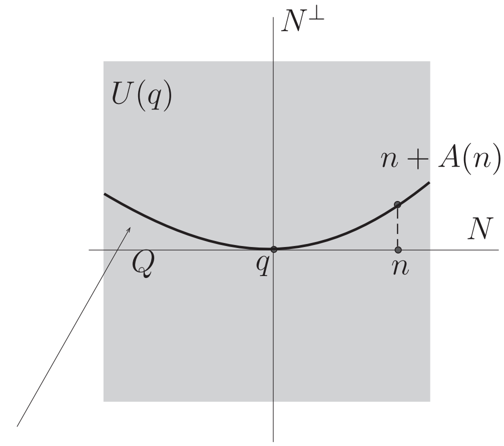
has its image in $U(q)$ and parameterizes all solutions p in $U(q)$ of the equation $f(p) = 0$.

(3) If $p = \Gamma(n_0) \in U(q)$ is a solution of $f(p) = 0$, then the map

$$\begin{aligned} \ker f'(q) &\to \ker f'(p) \\ \delta n &\mapsto \delta n + DA(n_0)\delta n \end{aligned}$$

is a linear isomorphism.

The map Γ *is called a* **good parametrization of the zero set near q**.



$\{p \in U(q) | \, f(p) = 0\} = \Gamma(Q).$

Every solution q of the above equation $f(q) = 0$ is smooth since f is assumed to be regularizing. The kernel $N = \ker f'(q)$ at the solution is finite dimensional since f is linearized Fredholm on the solution set and can be identified with the tangent space of the solution set at the point q.

The following propositions describe the "calculus of good parameterizations". We shall assume in the following that $b : K \to O$ is a local strong bundle and f an sc-smooth section which is regularizing and linearized sc-Fredholm at all solutions of $f = 0$.

Proposition 4.3. *If* $q \in O$ *is a zero of the section* f *and if there exists a good parametrization* $\Gamma : Q \to G \oplus E$ *around* q*, where* Q *is an open neighborhood of the origin in the kernel* N *of* $f'(q)$*, then given any zero* $q_0 = \Gamma(n_0)$ *of* f *in* $U(q)$*, there exists a good parametrization around* q_0*.*

Proof. By assumption, $f(q) = 0$ and there is a good parametrization

$$\Gamma : Q \to G \oplus E, \quad n \mapsto q + n + A(n).$$

In particular, we have an open neighborhood $U(q)$ in O so that every solution $p \in U(q)$ of $f(p) = 0$ lies in the image of Γ and at every solution $p \in U(q)$ the linearization $f'(p) : T_pO \to K_p$ is a surjective sc-Fredholm operator. Now pick $n_0 \in Q$ and let $q_0 = \Gamma(n_0)$. Then $f(q_0) = 0$. We shall construct a good

parametrization for the solutions set of f around q_0. Denote the projection from $G \oplus E = N \oplus N^{\perp}$ onto N by P. The kernel M of $f'(q_0)$ is given by

$$M = \{\delta n + DA(n_0)\delta n \mid \delta n \in N\}.$$

If $n = n_0 + h$, then

$$\begin{aligned}
\Gamma(n_0 + h) &= q + (n_0 + h) + A(n_0 + h) \\
&= q + (n_0 + h) + A(n_0) + DA(n_0)h \\
&\quad + [A(n_0 + h) - A(n_0) - DA(n_0)h] \\
&=: q_0 + (h + DA(n_0)h) + \Delta(h),
\end{aligned}$$

where $q_0 = \Gamma(n_0)$ and where $\Delta(0) = 0$ and $D\Delta(0) = 0$. Observing that $\delta n \mapsto \delta n + DA(n_0)\delta n$ is a linear isomorphism between N and M we take its inverse σ and define for $k \in M$ near 0 the map $B : M \to N^{\perp}$ by

$$B(k) := \Delta \circ \sigma(k).$$

Then

$$\Gamma \circ (n_0 + \sigma(k)) = q_0 + k + B(k)$$

and $M \oplus N^{\perp} = N \oplus N^{\perp}$. Hence the reparametrization $k \to \Gamma(n_0 + \sigma(k))$ of all the solutions near q_0 has the desired form

$$k \to q_0 + k + B(k),$$

where all the data are sc-smooth by construction. Now take an open neighborhood $U(q_0)$ in $U(q)$ and an open neighborhood Q' of $0 \in M$ so that the map $k \to q_0 + k + B(k)$ as a map from Q' into $U(q_0)$ parametrizes all solutions in $U(q_0)$. The proof of Proposition 4.3 is finished. ■

Next we study coordinate changes.

Proposition 4.4. *Assume that $U(q)$ is a small open neighborhood of our zero q and $\Phi : K|U(q) \to K'|U'(q')$ is a strong bundle isomorphism covering the diffeomorphism φ and denote by $g = \Phi_*(f)$ the push-forward of the section f. If the section f has a good parametrization Γ_0 near q, then there is a good parametrization Γ for the section g near $q' = \varphi(q)$ constructed from the good parametrization Γ_0 and the transformation Φ.*

Proof. Since $f(q) = 0$, we obtain for the linearized sections

$$g'(q') \circ T\varphi(q) = \Phi(q) \circ f'(q).$$

Hence the kernel N' of $g'(q')$ is the image under $T\varphi(q)$ of the kernel N of $f'(q)$. Consider the following composition defined on a sufficiently small open neighborhood Q' of 0 in N'

$$\Gamma : Q' \to G' \oplus E', \quad n' \mapsto \varphi(q + \sigma(n') + A(\sigma(n'))),$$

where $\sigma : N' \to N$ is the linear isomorphism obtained from the restriction of the inverse $T\varphi(q)^{-1}$. This map is well-defined since $q + n + A(n)$ belongs to $U(q)$ for n near 0 in N. Clearly, Γ parameterizes all solution of $g = 0$ near $q' = \Gamma(0)$. Using that $DA(0) = 0$, the linearisation of Γ at 0 is given by

$$D\Gamma(0) \cdot \delta n' = T\varphi(q) \circ \sigma(\delta n') = \delta n'.$$

Hence $D\Gamma(0) = 1$. Take a complement $(N')^{\perp}$ and consider the associated projection $P : G' \oplus E' = N' \oplus (N')^{\perp} \to N'$. Then the map $\tau : Q' \to N'$, defined by

$$\tau(n') := P(\Gamma(n') - q'),$$

is a local isomorphism near 0 preserving 0 whose linearization at 0 is the identity. Define on a perhaps smaller open neighborhood Q'' of $0 \in N'$ the mapping

$$A' : Q'' \to (N')^{\perp}, \quad n' \mapsto (I - P)(\Gamma \circ \tau^{-1}(n') - q').$$

Then

$$\begin{aligned} \Gamma \circ \tau^{-1}(n') &= q' + P(\Gamma \circ \tau^{-1}(n') - q') + (1 - P)(\Gamma \circ \tau^{-1}(n') - q') \\ &= q' + n' + A'(n'). \end{aligned}$$

In other words, using the good parametrization $n \mapsto q + n + A(n)$ of the zero set of the section f near q, we can build a good parametrization $n' \mapsto q' + n' + A'(n')$ of the zero set of the push-forward section $\Phi_* f$ near q' by only using the parametrization for f and data coming from the transformation Φ. ■

Next we study the smooth compatibility of good parameterizations. It is useful to recal from [12] that an sc-smooth map between open sets of smooth inite dimensional spaces is a C^∞-map in the familiar sense of calculus.

Proposition 4.5. *Let q_1 and q_2 be zeros of the section f and let N_1 and N_2 denote the kernels of the linearizations $f'(q_1)$ and $f'(q_2)$. Assume that*

$$\Gamma_i : Q_i \to N_i^{\perp}, \quad n \mapsto q_i + n + A_i(n)$$

are good parameterizations of the zero set of the section f near q_i for $i = 1, 2$. If $q \in U(q_1) \cap U(q_2)$ and $\Gamma_1(n_1) = \Gamma_2(n_2) = q$, then the local transition map $\Gamma_1^{-1} \circ \Gamma_2$ between open sets of smooth finite dimensional vector spaces defined near the point n_2 is of class C^{∞}.

Proof. By the properties of good parametrizations, solutions of $f(x) = 0$ near q are parametrized by the maps Γ_1 and Γ_2. Hence for every n close to n_2 in N_2 there exists a unique $\sigma(n)$ close to n_1 in N_1 satisfying

$$q_1 + \sigma(n) + A_1(\sigma(n)) = q_2 + n + A_2(n).$$

Applying the projection $P : Q \oplus E = N_1 \oplus N_1^{\perp} \to N_1$, we find

$$\sigma(n) = P(q_2 - q_1 + n + A_2(n)).$$

The map $n \mapsto \sigma(n)$ defined for n close to n_2 in N_2 into N_1 is sc-smooth as the composition of sc-smooth maps. Hence it is of class C^{∞} since N_1 and N_2 are smooth finite dimensional vector spaces. ■

4.2 Local Solutions Sets in the Interior Case

Our next aim is the proof of the existence of good parametrizations for the zero set of Fredholm sections as introduced in Definition 3.6. We recall from Proposition 3.8 that the linearization of a Fredholm section is an sc-Fredholm operator. But, as already pointed out, the mere requirement of the section to be linearized Fredholm with surjective linearization is not sufficient for the existence of good parametrizations.

Theorem 4.6. *Assume that f is a Fredholm section of the fillable strong M-polyfold bundle $b : K \to O$. Assume that the set O has no boundary. If the smooth point $q \in O$ is a solution of $f(q) = 0$ and if the linearization at this point,*

$$f'(q) : T_qO \to K_q$$

is surjective, then there exists a good parametrization of the solution set $\{f = 0\}$ near q.

In the proof of this theorem we shall make repeated use of the following result applied to various filled versions of the germ section.

Proposition 4.7. *Let O be an open neighborhood of 0 in some sc-Banach space $\mathbb{R}^n \oplus W$ and let $f : O \to \mathbb{R}^N \oplus W$ be an sc-smooth map satisfying $f(0) = 0$. We assume that the following holds.*

- *The map f is regularizing and its linearization $Df(0)$ at the smooth point $q = 0 \in O$ is an sc-Fredholm operator.*
- *The linearization $Df(0) : \mathbb{R}^n \oplus W \to \mathbb{R}^N \oplus W$ is surjective.*
- *With the projection $P : \mathbb{R}^N \oplus W \to W$ and writing an element in O as (v, w), the sc-smooth map*

  $$O \to W, \quad (v, w) \mapsto Pf(v, w)$$

  is an sc^0-contraction germ in the sense of Definition 2.1. In particular,

  $$Pf(v, w) = w - B(v, w)$$

  where B is a contraction in w on every level for the points (v, w) sufficiently close to $(0, 0)$ (the notion of close depending on the level m).

Let N be the kernel of $Df(0)$ and let $N^\perp$ be an sc-complement of N so that $N \oplus N^\perp = \mathbb{R}^n \oplus W$. Then there exists an open neighborhood Q of 0 in N, an open neighborhood U of 0 in O and a C^1-map

$$A : Q \to N^\perp, \quad n \mapsto A(n)$$

so that, with $\Gamma(n) = n + A(n)$, the following statements hold.

(1) *$A(0) = 0$ and $DA(0) = 0$ and for every $n \in Q$ the point $\Gamma(n)$ is smooth and solves the equation $f(\Gamma(n)) = 0$.*

(2) *Every solution of $f(p) = 0$ with $p \in U$ has the form $p = n + A(n)$ for a unique $n \in Q$.*

(3) *For every given pair $j, m \in \mathbb{N}$, there exists an open neighborhood $Q^{j,m} \subset Q$ of 0 in N so that*

  $$\Gamma : Q^{m,j} \to \mathbb{R}^n \oplus W_m$$

  is of class C^j.

(4) At every $q = \Gamma(n)$ with $n \in Q$, the linearization $Df(q)$ is surjective. Moreover, the projection $N \oplus N^\perp \to N$ induces an isomorphism ker $Df(q) \to$ ker $Df(0)$ between the kernels of the linearizations.

It is useful to recall from [12] that if an sc^0-map $f : U \subset E \to F$ on the open subset U of the sc-Banach space E to the sc-Banach space F is of class sc^k, then $f : U_{m+k} := U \cap E_m \to F_m$ is of class C^k for every $m \geq 0$. On the other hand, if for the sc^0-map $f : U \subset E \to F$ the induced maps $f : U_{m+k} \to F_m$ are of class C^{k+1} for every $m, k \geq 0$, then f is sc-smooth.

Note that the proposition guarantees the sc-smoothness of the map $n \mapsto A(n)$ at the distinguished point 0 only and not at other points in Q.

Proof of Proposition 4.7 First, by our results about contraction germs, there exists an open neighborhood V of 0 in $\mathbb{R}^n$ and a continuous map $\delta : V \to W$ satisfying $\delta(0) = 0$ and

$$Pf(v, \delta(v)) = 0.$$

By the regularizing property of the section f we conclude that $\delta(v)$ is a smooth point since $f(v, \delta(v)) = (1 - P)f(v, \delta(v)) \in \mathbb{R}^N$ is a smooth point. Hence

$$\delta : V \to W_\infty.$$

Since f is sc-smooth, the regularity results in the germ-implicit function theorem (Theorem 2.6) guarantee that there is a nested sequence of open neighborhoods of 0 in $\mathbb{R}^n$, say

$$V = V_0 \supset V_1 \supset \cdots \supset V_j \supset \cdots$$

so that the restrictions $\delta : V_j \to W_j$ are of class C^j for every $j \geq 0$. Now define $G : V \to \mathbb{R}^N$ by

$$G(v) := (1 - P)f(v, \delta(v)).$$

The map G restricted to V_j is of class C^j. Indeed, the map $V_j \to \mathbb{R}^n \oplus W_j$, $v \to (v, \delta(v))$ is C^j and f as a map from level j to level 0 is of class C^j. We need the following fact.

Lemma 4.8. *The map $DG(0) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^N)$ is surjective.*

Proof. The linearization of G at the point 0 is equal to

$$DG(0)(b) = (1 - P)Df(0) \cdot (b, \delta'(0)b) \tag{13}$$

for $b \in \mathbb{R}^n$. Given $(r, 0) \in \mathbb{R}^N \oplus W$ we can solve the equation

$$(r, 0) = Df(0)(b, h)$$

for $(b, h) \in \mathbb{R}^n \oplus W$, in view of the surjectivity of $Df(0)$. Equivalently, there exists $(b, h) \in \mathbb{R}^n \oplus W$ solving the two equations

$$\begin{aligned} r &= (1 - P)Df(0)(b, h) \\ 0 &= PDf(0)(b, h). \end{aligned}$$

Explicitly, the second equation is the following equation

$$0 = -D_1B(0) \cdot b + (1 - D_2B(0)) \cdot h \tag{14}$$

Recalling the proof of Theorem 2.3, the operator $1 - D_2B(0) : W \to W$ is a linear isomorphism. Therefore, given $b \in \mathbb{R}^n$ the solution h of the equation (14) is uniquely determined. On the other hand, linearizing $Pf(a, \delta(a)) = 0$ at the point $a = 0$ leads to $PDf(0)(b, \delta'(0)b) = 0$ for all $b \in \mathbb{R}^n$. Hence, by uniqueness $h = \delta'(0)b$, so that the linear map $DG(0)$ in (13) is indeed surjective. ■

Continuing with the proof of Proposition 4.7, we denote the kernel of $DG(0)$ by C and take its orthogonal complement $C^\perp$ in $\mathbb{R}^n$, so that

$$C \oplus C^\perp = \mathbb{R}^n.$$

Then $G : C \oplus C^\perp \to \mathbb{R}^N$ becomes a function of two variables, $G(c_1, c_2) = G(c_1 + c_2)$ for which $D_1G(0) = 0$ while $D_2G(0) \in \mathcal{L}(C^\perp, \mathbb{R}^N)$ is an isomorphism. By the implicit function theorem there exists a unique map $c : C \mapsto C^\perp$ solving

$$G(r + c(r)) = 0 \tag{15}$$

for r near 0 and satisfying

$$c(0) = 0, \quad Dc(0) = 0.$$

Moreover, given any $j \geq 1$, there is an open neighborhood U of the origin in C such that $c \in C^j(U, C^\perp)$. Summarizing we have demonstrated so far, that all solutions $(v, w) \in \mathbb{R}^n \oplus W$ of $f(v, w) = 0$ near the origin are represented by

$$f(r + c(r), \delta(r + c(r))) = 0$$

for r in an open neighborhood of 0 in $C = \ker DG(0) \subset \mathbb{R}^n$. We introduce the function $\beta : C \to \mathbb{R}^n \oplus W$ defined near 0 by

$$\beta(r) = (r + c(r), \delta(r + c(r))).$$

Then $f(\beta(r)) = 0$ for r near 0 in C. The function β satisfies $\beta(0) = 0$ and, of course, for every level m and every integer $j \geq 1$, there exists an open neighborhood U of $0 \in C$ such that if $r \in U$, then $\beta(r) \in \mathbb{R}^n \oplus W_m$ and $\beta \in C^j(U, \mathbb{R}^n \oplus W_m)$. Moreover, by the regularizing property of the section f, we know that all points $\beta(r)$ are smooth. In order to represent the solution set as a graph over the kernel of the linearized equation at 0 we introduce

$$N := \ker(Df(0)). \tag{16}$$

We have an sc-splitting

$$N \oplus N^{\perp} = \mathbb{R}^n \oplus W.$$

By construction, the image of the linearization $D\beta(0) \in \mathcal{L}(C, \mathbb{R}^n \oplus W)$ is equal to the kernel of $Df(0)$ and $D\beta(0) : C \to N$ is a linear isomorphism. We define the map $\alpha : N \to \mathbb{R}^n \oplus W$ near $0 \in N$ by

$$\alpha(n) = \beta(D\beta(0)^{-1} \cdot n).$$

The solution set is now parametrized by α so that $f(\alpha(n)) = 0$ for n near $0 \in N$. Let $P : N \oplus N^{\perp} \to N$ be the projection along $N^{\perp}$ and consider the map $P \circ \alpha : N \to N$ near 0. Since $D(P \circ \alpha)(0) = 1$, it is a local diffeomorphism leaving the origin fixed. We denote by γ the inverse of this local diffeomorphism satisfying $P \circ \alpha(\gamma(n)) = n$ for all small n. So,

$$\begin{aligned} \alpha(\gamma(n)) &= P \circ \alpha(\gamma(n)) + (1 - P)\alpha(\gamma(n)) \\ &= n + (1 - P)\alpha(\gamma(n)). \end{aligned}$$

Define the map

$$A : N \to N^{\perp}$$

near the origin in N by $A(n) = (1-P)\alpha(\gamma(n))$. Then $A(0) = 0$ and $DA(0) = 0$ since the image of $\alpha'(0)$ is equal to the kernel N. Moreover, $\alpha(\gamma(n)) = n + A(n)$ and

$$f(n + A(n)) = 0.$$

In addition, given any level m and any integer j there exists an open neighborhood U of 0 in N such that if $n \in U$, then $A(n) \in N^{\perp} \cap (\mathbb{R}^n \oplus W_m)$

and $A \in C^j(U, \mathbb{R}^n \oplus W_m)$. We have demonstrated that the solution set of $f(v, w) = 0$ near the transversal point 0 is represented as a graph over the kernel of the linearized map $Df(0)$ of a function which is sc-smooth at the point 0. In order to complete the proof of Proposition 4.7 it remains to verify property (4) of the proposition for a suitable Q. This is the next lemma.

Lemma 4.9. *If n_0 is small enough and $q_0 = \Gamma(n_0)$, then the linearization $Df(q_0) : \mathbb{R}^n \oplus W \to \mathbb{R}^N \oplus W$ is surjective. Moreover,*

$$\mathit{dim}\ [\mathit{ker}\ Df(q_0)] = \mathit{dim}\ [\mathit{ker}\ Df(0)] = \mathit{dim}\ N.$$

Setting $N' = \mathit{ker}\ Df(q_0)$, the natural projection $P : N \oplus N^{\perp} = \mathbb{R}^n \oplus W \to N$ induces an isomorphism $P|N' : N' \to N$.

Proof. Consider the linearized operator

$$Df(q_0) : \mathbb{R}^n \oplus W \to \mathbb{R}^N \oplus W \tag{17}$$

at the solution point

$$q_0 = (a, \delta(a)) = \Gamma(n_0).$$

In order to prove surjectivity we consider the equation $Df(q_0)(b, h) = (r, w)$. In matrix notation we interpret this equation as a system

$$\begin{bmatrix} A & A_1 \\ A_2 & A_3 \end{bmatrix} \cdot \begin{bmatrix} h \\ b \end{bmatrix} = \begin{bmatrix} w \\ r \end{bmatrix} \tag{18}$$

with the linear operators

$$\begin{aligned} Ah &= (1 - D_2 B(q_0)) \cdot h \\ A_1 b &= D_1 B(q_0) \cdot b \\ A_2 h &= (1 - P) D_2 f(q_0) \cdot h \\ A_3 b &= (1 - P) D_1 f(q_0) \cdot b \end{aligned}$$

where $(b, h) \in \mathbb{R}^n \oplus W$ and $(r, w) \in \mathbb{R}^N \oplus W$. Since the point q_0 is smooth, the linear operator $A : W \to W$ is an sc-isomorphism in view of the proof of Theorem 2.3. Hence the equation (18) for (h, b) becomes

$$\begin{aligned} h &= A^{-1} w - A^{-1} A_1 b \\ (A_2 A^{-1} A_1 - A_3) b &= A_2 A^{-1} w - r. \end{aligned} \tag{19}$$

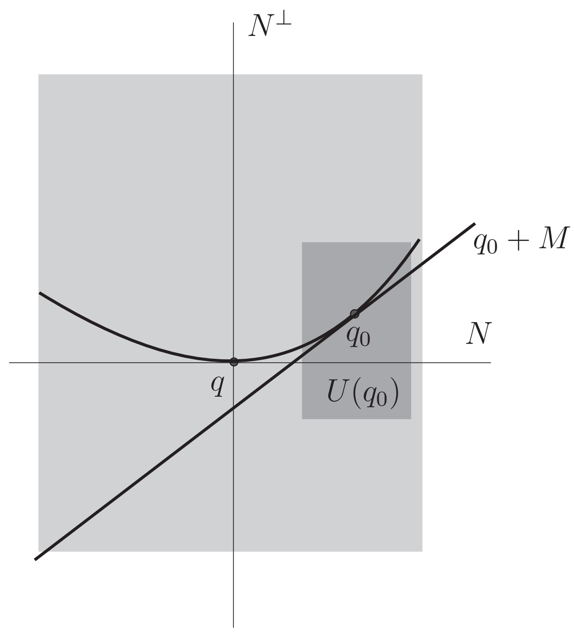
We abbreviate the continuous family of matrices

$$M(a) := [A_2 A^{-1} A_1 - A_3] \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^N).$$

By assumption, $Df(0)$ is surjective. Therefore, if $a = 0$, then the matrix $M(0)$ is surjective. Consequently, for small a the matrix $M(a)$ is also surjective and so is the linear operator $Df(a, \delta(a))$. Choosing $w = 0$ and $r = 0$ in (19), the kernel of $Df(q_0)$ is determined by the two equations

$$\begin{aligned} h &= A^{-1} A_1 b \\ [A_2 A^{-1} A_1 - A_3] b &= 0. \end{aligned}$$

Consequently, the kernel is determined by the kernel of the matrix $M(a)$. Recalling that $N' = \ker Df(q_0)$ and $N = \ker Df(0)$ we conclude from the surjectivity of $M(a)$ for small a that $\dim N' = \dim N$. In addition, the natural projection $P : N \oplus N^{\perp} = \mathbb{R}^n \oplus W \to N$ induces the isomorphism $P|N' : N' \to N$ and the lemma follows and therefore also Proposition 4.7 is proved. ■



Proof of Theorem 4.6. After all these preparations we are ready to prove Theorem 4.6 about the Fredholm section f of the fillable local M-polyfold bundle $b : K \to O$. By assumption, the distinguished smooth point $q \in O$ solves the equation $f(q) = 0$ and the linearized map $f'(q)$ is surjective. In view of the Fredholm property of the section f as defined in Definition 3.6 and the surjectivity of $f'(0)$, there exist coordinates in which q corresponds to 0 in $\mathbb{R}^n \oplus W$ and in which the the filled version $\bar{\mathbf{f}} : \mathbb{R}^n \oplus W \to \mathbb{R}^N \oplus W$ of the section f, which is defined in an open neighborhood of 0, meets the assumptions of Proposition 4.7. We have used here the fact proved in section 3.2 that the linearization $D\bar{\mathbf{f}}(0)$ of every filled version is surjective if and only if $f'(q)$ is surjective. From section 3.2 we know that the zero set of $\bar{\mathbf{f}}$ is equal to the zero set of the local representation of f in our coordinates. In view of the discussion in section 3.2 and the invariance of good parametrizations (Proposition 4.4) under coordinate transformations, it is sufficient to establish a good parametrization for the filled section $\bar{\mathbf{f}}$ near 0.

From Proposition 4.7 we deduce, using the notation of the proposition (but replacing f by $\bar{\mathbf{f}}$), that there exist an open neighborhood Q of 0 in the kernel N of $\bar{\mathbf{f}}$, an open neighborhood $U(0)$ in $\mathbb{R}^n \oplus W$ and a mapping

$$\Gamma : Q \to U(0), \quad n \mapsto n + A(n)$$

of class C^1 satisfying $A(0) = 0$, $DA(0) = 0$ and $A(n) \in N^\perp$ and having the following additional properties.

- $\bar{\mathbf{f}}(\Gamma(n)) = 0$ for all $n \in Q$, and every solution $p \in U(0)$ of $\bar{\mathbf{f}}(p) = 0$ lies in the image of the map Γ.
- For every given pair $j, m \in \mathbb{N}$, there exists an open neighborhood $Q^{j,m} \subset Q$ of 0 in N so that

  $$\Gamma : Q^{m,j} \to \mathbb{R}^n \oplus W_m$$

  is of class C^j.
- At every point $q = \Gamma(n)$ with $n \in Q$, the linearization $D\bar{\mathbf{f}}(q)$ is surjective, and if $N' = \ker D\mathbf{f}(q)$, then the projection $P : N \oplus N^\perp \to N$ induces an isomorphism $P|N' : N' \to N$ between the kernels.

We conclude, in particular, that the map $\Gamma : Q \to U(0)$ is sc-smooth at the distinguished point $0 \in Q$ and we have to prove that Γ is sc-smooth at every

other point of Q, since all the other properties of a good parametrization near 0 already hold true. So, we choose $n_0 \in Q$ and prove that the map Γ is sc-smooth at this point. We already know that $q_0 = \Gamma(n_0)$ solves $\bar{\mathbf{f}}(q_0) = 0$ and that the linearization $D\bar{\mathbf{f}}(q_0)$ is surjective. Consequently, the linearization $f'(q_0)$ is surjective and therefore also the linearization of every other filled version at the point q_0 is surjective. Now we observe that the section $\bar{\mathbf{f}}$ is a filled version with respect to the distinguished point $q = 0$ and not necessarily with respect to the point q_0. Therefore, we cannot conclude that $\bar{\mathbf{f}}$ possesses all the above nice properties near q. However, by the definition of a Fredholm section, after some coordinate change perhaps using some other filler and an additional change of coordinates, there exists a contraction normal form near q_0. In view of the surjectivity of the linearization at the point q_0, we can apply Proposition 4.7 to this new situation. Then going back with all these coordinate transformations we obtain a map

$$\begin{aligned} \Theta &: Q' \to U(q_0) \subset U(0) \\ \Theta(m) &= q_0 + m + B(m), \end{aligned}$$

where Q' is an open neighborhood of 0 in the kernel M of $D\bar{\mathbf{f}}(q_0)$. Moreover, $B(m)$ lies in some sc-complement $M^\perp$. Near $m = 0$, the map B has the same properties as the map A, in particular, B is sc-smooth at $m = 0$ in M. We know that $\Theta(0) = q_0 = \Gamma(n_0)$ and that for every m near 0, there exists a unique n close to n_0 solving the equation $\Theta(m) = \Gamma(n)$. More explicitly,

$$q_0 + m + B(m) = n + A(n). \tag{20}$$

If $P : \mathbb{R}^n \oplus W = N \oplus N^\perp \to N$ is the projection along $N^\perp$, we define the map $\alpha : M \to N$ near $m = 0$ by

$$\alpha(m) = P(q_0 + m + B(m)).$$

It satisfies

$$\alpha(0) = P(q_0) = P(n_0 + A(n_0)) = n_0.$$

The linearization $D\alpha(0)$ is equal to the projection $P : M \to N$ which is an isomorphism since M is a graph of a linear map $N \to N^\perp$. Moreover, the map α has arbitrarily high differentiability near $m = 0$. From (20) we conclude

$$n = \alpha(m)$$

and taking the inverse near $m = 0$, we have $m = \alpha^{-1}(n)$. Using (20) again, we obtain $n + A(n) = q_0 + \alpha^{-1}(n) + B(\alpha^{-1}(n))$, and applying $1 - P$ to both sides we arrive at

$$A(n) = (1 - P)(q_0 + \alpha^{-1}(n) + B(\alpha^{-1}(n))).$$

The inverse map α^{-1} possesses for n near n_0 arbitrary high differentiability and we conclude that the map A is sc-smooth at the point n_0. The argument applies to every $n_0 \in Q$ and we conclude that the map $\Gamma : Q \to N^{\perp}$ is sc-smooth, hence a good parametrization of the filled version $\bar{\mathbf{f}}$ of f near 0. The proof of Theorem 4.6 is complete. ■

4.3 Good Parameterizations in the Boundary Case

We consider the fillable local strong M-polyfold bundle

$$b : K \to O$$

where O is an open subset in the splicing core $K^{\mathcal{S}}$. This time the splicing $\mathcal{S} = (\pi, E, V)$ has the property that the parameter set V is a relatively open subset of a partial quadrant $C \subset G$. We may assume that C is of the form

$$C = [0, \infty)^k \oplus W$$

for some sc-Banach space W and $0 \in V$. We study a section

$$f : O \to K$$

of the bundle b and assume that the boundary point $0 \in \partial O$ is a solution of $f(q) = 0$ and that the linearization

$$f'(0) : T_0 O \to K_0$$

is surjective. The aim is to describe the solution set $\{f = 0\}$ near $0 \in \partial O$ under the assumption that f is a Fredholm section. Since 0 is a boundary point we are confronted with some subtleties. If, for example, the kernel N of $f'(0)$ is in a bad position to the boundary, there might be no solutions of $f = 0$ apart from 0, even if the Fredholm index is large and the linearization $f'(0)$ is surjective.

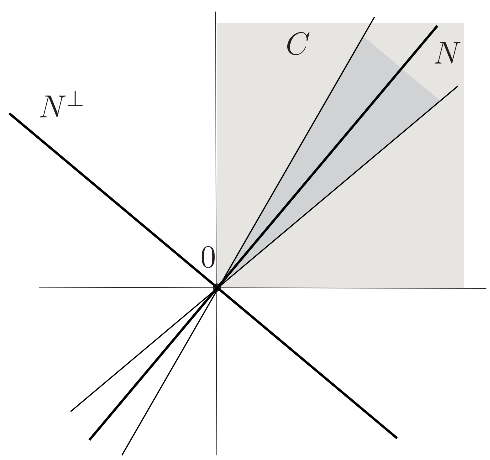
Definition 4.10. *Let C be a partial quadrant in the sc-Banach space E.*

- *The closed linear subspace N of E is in* **good position to the partial quadrant** *C of E, if $N \cap C$ has a nonempty interior in N and if there exist an sc-complement $N^{\perp}$ of N in E and a positive constant c, so that for a point $(n, m) \in N \oplus_{sc} N^{\perp}$ satisfying*

$$\|m\|_E \leq c \cdot \|n\|_E$$

the statements $n + m \in C$ and $n \in C$ are equivalent. We call $N^{\perp}$ a **good complement***.*

- *A finite-dimensional subspace N of E is called* **neat with respect to the partial quadrant** *C if there exists an sc-complement $N^{\perp}$ of N in E satisfying $N^{\perp} \subset C$.*



The choice of the complement $N^{\perp}$ is important, i.e. if the defining condition holds for one choice it does not need to hold for another choice (at least in higher dimensions). This definition is very practical for the local theory. Observe that if $N \subset \mathbb{R}^n \oplus W$ is a neat subspace with respect to the partial quadrant $C = [0, \infty)^n \oplus W$, then necessarily $\dim N \geq n$. Moreover, the following holds.

Proposition 4.11. *A neat subspace is in good position to the partial quadrant.*

The proof of Proposition 4.11 is carried out in Appendix 6 (Proposition 6.2).

Now assume that q is a smooth point in the boundary of the open subset O of the splicing core $K^{\mathcal{S}}$. Then $q = (v, e) \in V \oplus E$ satisfies $\pi_v(e) = e$. Recall that

$$V \subset C \subset \mathbb{R}^k \oplus W = G.$$

The tangent space T_qO is the space

$$T_qO = \{(\delta v, \delta e) \in G \oplus E |\, \delta e = \pi_v \delta e + D_v(\pi_v e) \cdot \delta v\}.$$

With every tangent vector $(\delta v, \delta e) \in T_qO$ we can associate the continuous path

$$t \to (v + t\delta v, \pi_{v+t\delta v}(e + t\delta e))$$

in $G \oplus E$. The path starts at $q = (v, e)$ at time $t = 0$ and belongs to O as long as $v + t\delta v$ is close to v and belongs to V. Denote by $C_v \subset G$ the collection of all tangent vectors δv so that $v + t\delta v$ belongs to V for $t > 0$ small enough. The set C_v constitutes a partial quadrant in the space G. Hence $C_v \oplus \ker(1 - \pi_v)$ is a partial quadrant in the space $G \oplus \ker(1 - \pi_v)$.

Taking the derivative of the identity $\pi_v^2(e) = \pi_v(e)$ in the variable v one obtains $D_v\pi_v(\pi_v(e)) \cdot \delta v + \pi_v(D_v\pi_v(e) \cdot \delta v) = D_v\pi_v(e) \cdot \delta v$ and one concludes, in view of $\pi_v(e) = e$, that $\pi_v(D_v\pi_v(e) \cdot \delta v) = 0$. Assuming, in addition, that $\pi_v(\delta e) = \delta e$ one finds that $\delta e + D_v\pi_v(e) \cdot \delta v = \pi_v(\delta e + D_v\pi_v(e) \cdot \delta v) + D_v\pi_v(e) \cdot \delta v$. Thus,

$$(\delta v, \delta e + D_v\pi_v(e) \cdot \delta v) \in T_qO$$

for all $\delta v \in G$ and $\delta e \in \ker(1 - \pi_v)$, where $q = (v, e)$ satisfies $\pi_v(e) = e$. The map $A : G \oplus E \to G \oplus E$, defined by

$$A : (\delta v, \delta e) \mapsto \big(\delta v, \delta e + D_v\pi_v(e) \cdot \delta v\big),$$

maps the space $G \oplus \ker(1 - \pi_v)$ sc-isomorphically onto the tangent space T_qO and consequently the partial quadrant $C_v \oplus \ker(1 - \pi_v)$ onto the partial quadrant $C_q := A(C_v \oplus \ker(1 - \pi_v))$ in the tangent space T_qO.

Definition 4.12. *The partial quadrant $C_q \subset T_qO$ is called the* **partial tangent quadrant** *at the smooth point $q \in O$.*

The following result is readily verified.

Lemma 4.13. *If an sc-diffeomorphism $\varphi : O \to O'$ maps the smooth point $q \in O$ to the smooth point $q' \in O'$, then its tangent map $T\varphi(q)$ at q maps C_q to $C'_{q'}$. If $N \subset T_qO$ is a finite-dimensional smooth subspace which is in good position to C_q or which is neat with respect to C_q, then its image $T\varphi(q)N$ has the same property with respect to $C'_{q'}$.*

The previous lemma allows to introduce the following intrinsic notions.

Definition 4.14. *Let x be a smooth point in a M-polyfold X and let $\varphi : (U(x), x) \to (O, q)$ be a local chart around x.*

- *The* **partial tangent quadrant** *Q_x in T_xX is the subset which is mapped under $T\varphi(x)$ onto the partial tangent quadrant C_q of T_qO.*
- *A smooth finite-dimensional subspace N of T_xX is in* **good position** *to the corner structure of X at x (or in* **good position** *to the partial tangent quadrant Q_x) if this is true for the subspace $T\varphi(x)N$ with respect to C_q in T_qO.*
- *A smooth finite-dimensional subspace N of T_xX is* **neat** *with respect to the corner structure of X at x (or* **neat** *with respect to the partial tangent quadrant Q_x), if the same holds for $T\varphi(x)N$ with respect to C_q in T_qO.*

Remark 4.15. *Since an sc-isomorphism $A : G \oplus E \to G \oplus E$ introduced above leaves the linear spaces $\{0\} \oplus \ker \pi_v$ invariant, the tangent spaces T_qO at the smooth point $q = (v, e) \in O$ has $\ker \pi_v$ as natural sc-complement in $T_q(V \oplus E)$ so that*

$$G \oplus E = T_q(V \oplus E) = T_qO \oplus \ker \pi_v.$$

If $N \subset T_{(v,e)}O$ is in good position to the corner structure and $N^\perp \subset T_qO$ is a good complement of N in T_qO, then $N^\perp \oplus \ker \pi_v$ is a good complement of N in $G \oplus E$. In particular, $N \subset G \oplus E$ is in good position to the partial quadrant $C \oplus E$ in $G \oplus E$.

Next we define what it means that a Fredholm section is in a good position at a solution q of $f(q) = 0$.

Definition 4.16. *Consider a Fredholm section f of the fillable strong M-polyfold bundle $p : Y \to X$ and assume that $f(q) = 0$. Then f is said to be in* **good position to the corner structure** *at the point x provided*

- *The linearization $f'(x)$ is surjective.*
- *The kernel N of $f'(x)$ is in good position to the corner structure of X at x.*

The previous concept of a good parametrization, adapted to the situation with boundary, is useful to describe the solution set near the boundary.

Definition 4.17. *Consider a local strong bundle $b : K \to O$ where O has no boundary. Assume that the section f of the bundle b is regularizing and at every solution $p \in O$ of $f(p) = 0$ the section f is linearized Fredholm and its linearization $f'(p) : T_pO \to K_p$ is surjective. Let $q \in \partial O$ be a solution of $f(q) = 0$ at the boundary of O and assume that the kernel N of the linearization $f'(q) : T_qO \to K_q$ is in good position to the partial tangent quadrant $C_q \subset T_qO$. Then the section f is said to have a* **good parametrization of the solution set** *$\{f = 0\}$ near q, if there exist an open neighborhood Q of 0 in $N \cap C_q$ (which in view of Proposition 6.1 is a partial quadrant in N), an open neighborhood $U(q)$ of q in O, and an sc-smooth map*

$$A : Q \to N^{\perp}$$

into some good complement $N^{\perp}$ of N in $G \oplus E$ such that the following properties are satisfied.

(1) $A(0) = 0$ and $DA(0) = 0$.

(2) The map

$$\Gamma : Q \to G \oplus E, \quad n \mapsto q + n + A(n)$$

has its image in $U(q)$ and parameterizes all solutions $p \in U(q)$ of the equation $f(p) = 0$.

(3) At every solution $p \in U(q)$ of $f(p) = 0$, the map

$$\begin{gathered} \ker f'(q) \to \ker f'(p) \\ \delta n \mapsto \delta n + DA(n_0) \cdot \delta n \end{gathered}$$

is a linear isomorphism, where $p = q + n_0 + A(n_0)$.

The map Γ is called a **good parametrization of the zero set** *$\{f = 0\}$ near the zero $q \in \partial O$.*

The calculus of good parameterizations in the boundary case is similar to the previously discussed interior case. Let us observe that, in view of Proposition 6.1, the set $C_q \cap N$ is a partial quadrant in N. Hence a good parametrization is automatically parameterizing locally a manifold with boundary with corners.

4.4 Local Solutions Sets in the Boundary Case

The following main result of this section guarantees the existence of good parameterizations of solution sets near boundary points.

Theorem 4.18. *Let $b : K \to O$ be a fillable local strong M-polyfold bundle and $f : O \to K$ a Fredholm section of the bundle b. Assume that q is a smooth point in ∂O solving the equation $f(q) = 0$. If the section f is in good position to the corner structure at the point q (in the sense of Definition 4.16), then there exists a good parametrization of the solution set $\{f = 0\}$ near q.*

Proof. The proof goes along the same lines as the proof of Theorem 4.7 in the case without boundaries with some minor modifications we describe next. Recall that the Fredholm property tells us that, after a possible change of coordinates, there is a filler so that after another change of coordinates we have a contraction germ. Hence, without loss of generality we assume that $q = 0 \in \partial O$ and that f is already filled and

$$f : O \subset ([0, \infty)^k \oplus \mathbb{R}^{n-k}) \oplus W \to \mathbb{R}^N \oplus W.$$

Since we started with a section which at the point q is in good position to the corner structure of O, the filled section is, by Remark 4.15, in good position to the corner structure at 0. Moreover, if $P : \mathbb{R}^N \oplus W \to W$ is the canonical projection, we have the representation

$$Pf(v, w) = w - B(v, w),$$

where v is near 0 in the partial quadrant $[0, \infty)^k \oplus \mathbb{R}^{n-k}$ of the space $\mathbb{R}^n$, and where $w \in W$. The contraction property holds near $(0, 0)$ where the notion of near depends on the level m in W. Thus, in view of Theorem 2.6, there exists a unique map

$$v \to \delta(v)$$

from a neighborhood of 0 in $[0, \infty)^k \oplus \mathbb{R}^{n-k}$ into W solving the equation

$$Pf(v, \delta(v)) = 0$$

and satisfying $\delta(0) = 0$. The map δ has the same properties as in the case without boundary. Introduce the mapping $g : [0, \infty)^k \oplus \mathbb{R}^{n-k} \to \mathbb{R}^N$ defined near 0 by

$$g(v) := (1 - P)f(v, \delta(v)).$$

The map g has, as in in the case without boundary, a surjective linearization at 0 and the kernel N' of $g'(0) : \mathbb{R}^n \to \mathbb{R}^N$ is mapped by

$$h \mapsto (h, \delta'(0)h)$$

isomorphically onto the kernel N of the linearization $f'(0)$. Abbreviate the partial quadrants

$$\begin{aligned} C &:= [0,\infty)^k \oplus \mathbb{R}^{n-k} \oplus W \quad \text{in } \mathbb{R}^n \oplus W \\ C' &:= [0,\infty)^k \oplus \mathbb{R}^{n-k} \quad \text{in } \mathbb{R}^n. \end{aligned}$$

Since N is in good position to the partial quadrant $[0,\infty)^k \oplus \mathbb{R}^{n-k} \oplus W$ we can take a good complement $N^\perp$ of N in $\mathbb{R}^n \oplus W$. Consider the linear sc-isomorphism

$$\mathbb{R}^n \oplus W \to \mathbb{R}^n \oplus W, \quad (h,k) \mapsto (h, k + \delta'(0)h).$$

It maps C onto C. Identifying the kernel $N' \subset \mathbb{R}^n$ with $N' \oplus \{0\}$ in $\mathbb{R}^n \oplus W$, we see that he image of N' under the above map is the kernel N. Hence the preimage $(N')^\perp$ of the good complement $N^\perp$ of N is a good complement of N' in $\mathbb{R}^n \oplus W$. In particular, $M = (N')^\perp \cap \mathbb{R}^n \oplus \{0\}$ is a good complement of N' in $\mathbb{R}^n$.

Since the kernel N' is in good position to C' in $\mathbb{R}^n$, we see that $N' \cap C'$ is a partial quadrant in view of Proposition 6.1 in Appendix. By the finite-dimensional implicit function theorem (one might first extend the map g to a C^1-map on an open neighborhood of 0 in $\mathbb{R}^n$ by Whitney's extension theorem), we obtain a solution germ of $g(r + c(r)) = 0$ where

$$r \mapsto r + c(r). \tag{21}$$

is a function from $N' \cap C'$ into $N' \oplus M = \mathbb{R}^n$ defined near $r = 0$. In particular, $c(r) \in M$, moreover, $c(0) = 0$ and $Dc(0) = 0$. As a side remark we observe that the function (21) is a good parametrization for our finite-dimensional problem in the sense of the above definition. Define the map

$$\begin{aligned} &\Theta : N' \cap C' \to N' \oplus M \oplus W \\ &\Theta(r) = r + c(r) + \delta(r + c(r)). \end{aligned}$$

Take the bijective map $\sigma : N' \cap C' \to N \cap C$ defined by $\sigma(r) = r + \delta'(0)r \in \mathbb{R}^n \oplus W$ and introduce the map

$$\begin{aligned} &\Gamma : N \cap C \to \mathbb{R}^n \oplus W \\ &\Gamma(n) = \Theta \circ \sigma^{-1}(n). \end{aligned}$$

Then (recall that $q = 0$) we can write

$$\Gamma(n) = q + n + A(n)$$

with $A(n) \in N^{\perp}$ and $A(0) = 0$ and $DA(0) = 0$. Moreover, $f(\Gamma(n)) = 0$, so that Γ parameterizes the solution set $\{f = 0\}$ near $q = 0$. Arguing as in the proof of Theorem 4.7 one sees that the map Γ is sc-smooth. Therefore, Γ is a good parametrization and the proof of Theorem 4.18 is complete. ■

The above existence proofs of good parametrizations of solution sets of Fredholm sections prompts the following concept of a finite dimensional submanifold of an M-polyfold. Such submanifolds have the structure of a manifold induced in a natural way.

Definition 4.19. *Let X be an M-polyfold and $M \subset X$ a subset equipped with the induced topology. The subset M is called a* **finite dimensional submanifold** *of X provided the following holds.*

- *The subset M lies in X_∞.*
- *At every point $m \in M$ there exists an M-polyfold chart*

  $$(U, \varphi, (\pi, E, V))$$

  where $m \in U \subset X$ and where $\varphi : U \to O$ is a homeomorphism satisfying $\varphi(m) = 0$, onto the open neighborhood O of 0 in the splicing core K associated with the sc-smooth splicing (π, E, V). Here V is an open neighborhood of 0 in a partial quadrant C of the sc-Banach space W. Moreover, there exists a finite-dimensional smooth linear subspace $N \subset W \oplus E$ in good position to C and a corresponding sc-complement $N^{\perp}$, an open neighborhood Q of $0 \in C \cap N$ and an sc-smooth map $A : Q \to N^{\perp}$ satisfying $A(0) = 0$, $DA(0) = 0$ so that the map

  $$\Gamma : Q \to W \oplus E : q \to q + A(q)$$

  has its image in O and the image of the composition $\Phi := \varphi^{-1} \circ \Gamma : Q \to U$ is equal to $M \cap U$.
- *The map $\Phi : Q \to M \cap U$ is a homeomorphism.*

We recall that $N \subset W \oplus E$ is a smooth subspace if $N \subset (W \oplus E)_\infty = W_\infty \oplus E_\infty$. The map $\Phi : Q \to U$ is called a **good parametrization** *of a neighborhood of $m \in M$ in M.*

In other words a subset $M \subset X$ of an M-polyfold X consisting of smooth points is a submanifold if for every $m \in M$ there is a good parametrization of an open neighborhood of m in M. The following proposition shows that the transition maps $\Phi \circ \Psi^{-1}$ defined by two good parameterizations Φ and Ψ are smooth, so the inverses of the good parametrizations define an atlas of smoothly compatible charts. Consequently, a finite dimensional submanifold is in a natural way a manifold with boundary with corners.

Proposition 4.20. *Any two parametrizations of a finite dimensional submanifold M of the M-polyfold X are smoothly compatible.*

Proof. Assume that $m_0 := \varphi^{-1}(q_0 + A(q_0)) = \psi^{-1}(p_0 + B(p_0))$ for two good parameterizations. Since both good parameterizations are local homeomorphisms onto an open neighborhood of m_0 in M, we obtain a local homeomorpism $O(p_0) \to O(q_0)$, $p \mapsto q(p)$, where the domain and codomain are relatively open neighborhoods in partial quadrants. We have

$$q(p) + A(q(p)) = \varphi \circ \psi^{-1}(p + B(p)).$$

Recall that $q(p) \in N$ and $A(q(p) \in N^{\perp}$, where $N \oplus N^{\perp} = W \oplus E$ is an sc-splitting. If $P : W \oplus E \to N$ is an sc-projection along $N^{\perp}$, then

$$q(p) = P(\varphi \circ \psi^{-1}(p + B(p)).$$

The map $p \mapsto q(p)$ is sc-smooth as a composition of sc-smooth maps. However, since the domain and codomain lie in finite dimensional smooth linear spaces, the map is of class C^{∞}. ■

Examples of finite dimensional submanifolds of M-polyfolds are the solution sets of Fredholm sections in the case of transversality. We would like to mention that also the strong finite dimensional submanifolds introduced in Definition 3.19 of [12] are finite dimensional submanifolds in the sense of Definition 4.19. The induced manifold structures are the same.

5 Global Fredholm Theory

This section is devoted to the Fredholm theory in M-polyfold bundles. Putting together the local studies of the previous section it will be proved, in particular, that the solution set $f^{-1}(0)$ of a proper Fredholm section f of a fillable

strong M-polyfold bundle $p: Y \to X$ carries in a natural way the structure of a smooth compact manifold with boundary with corners, provided at every point $q \in f^{-1}(0)$ the section f is in good position to the corner structure of X.

For the deeper study of Fredholm operators it is useful to introduce first some auxiliary concepts.

5.1 Mixed Convergence and Auxiliary Norms

We begin with a notion of convergence in the strong M-polyfold bundle $p: Y \to X$ called mixed convergence, referring to a mixture of strong convergence in the base space X and weak convergence in the level 1 fibers $Y_{0,1}$, assuming the fibers are reflexive sc-Banach spaces. These spaces are characterized by the property that bounded sequences have weakly convergent subsequences, a fact useful in compactness proofs. We first give the local definition in a strong M-polyfold bundle chart.

To do this we consider the local M-polyfold bundle

$$K \to O$$

where $K = K^{\mathcal{R}}$ is the splicing core

$$K = \{(v, e, u) \in O \oplus F \mid \rho_{(v,e)}(u) = u\}$$

of the strong bundle splicing $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$. Here F is an sc-Banach space and O is an open subset of the splicing core $K^{\mathcal{S}} = \{(v, e) \in V \oplus E \mid \pi_v(e) = e\}$ associated with the splicing $\mathcal{S} = (\pi, E, V)$ in which V is an open subset in the partial quadrant C of the sc-Banach space G.

Abbreviating the elements in O by $x = (v, e)$ we consider a sequence

$$(x_k, y_k) \in K_{0,1} = \{(x, y) \in O \oplus F_1 \mid \rho_x(y) = 0\}.$$

Definition 5.1 (**Mixed convergence**)**.** *The sequence $(x_k, y_k) \in K_{0,1}$ is called* **m-convergent** *to $(x, y) \in (G \oplus E)_0 \oplus F_1$ if*

$$\begin{aligned} &(1) \qquad x_k \to x \ \textit{in } O \\ &(2) \qquad y_k \rightharpoonup y \ \textit{in } F_1. \end{aligned}$$

Symbolically,

$$(x_k, y_k) \xrightarrow{m} (x, y).$$

The half arrow $\rightharpoonup$ denotes the weak convergence.

Lemma 5.2. *If* $(x_k, y_k) \in K_{0,1} \subset (G \oplus E)_0 \oplus F_1$ *and* $(x_k, y_k) \xrightarrow{m} (x, y)$*, then* $(x, y) \in K_{0,1}$.

Proof. By assumption, $x_k = (v_k, e_k)$ and $\pi_{v_k}(e_k) = e_k$. Moreover, $\rho_{x_k}(y_k) = y_k$. Since the inclusion map $\imath : F_1 \to F_0$ is a compact operator, one concludes from the weak convergence of $y_k \rightharpoonup y$ in F_1 that $y_k \to y$ in F_0. Since $x_k \to x = (v, e)$, it follows from the sc^0-continuity of the projections $(x, y) \mapsto \rho_x(y)$ from $O \oplus F$ into F that $\rho_x(y) = y$ in F_0. Since $y \in F_1$, one concludes $\rho_x(y) = y$ in F_1 so that $(x, y) \in K_{0,1}$ as claimed in the lemma. ∎

We shall demonstrate next that the concept of m-convergence is compatible with M-polybundle maps provided the fibers are reflexive spaces according to the following definition.

Definition 5.3. *An sc-Banach space* F *is called* **reflexive** *if all the spaces* F_m*,* $m \geq 0$*, of its filtration are reflexive Banach spaces.*

In order to verify that the notion of mixed convergence is invariant under M-polyfold chart transformations we take a strong M-polyfold bundle map

$$\Phi : K^{\mathcal{R}} \to K^{\mathcal{R}'}$$

between the splicing cores associated with the strong bundle splicings $\mathcal{R}$ and $\mathcal{R}'$. We assume that the underlying sc-Banach spaces F and F' are reflexive. Abbreviating $x = (v, e) \in O$, the map Φ is of the form

$$\Phi(x, y) = (\sigma(x), \varphi(x, y))$$

where $\sigma : O \to O'$ is an sc-diffeomorphism and φ linear in the fibers. Moreover, Φ induces two sc-diffeomorphisms $K^{\mathcal{R}}(i) \to K^{\mathcal{R}'}(i)$ for $i = 0$ and $i = 1$. In the following we shall drop the indication of the strong splicings and simply write $K = K^{\mathcal{R}}$ and $K' = K^{\mathcal{R}'}$.

We consider a sequence $(x_k, y_k) \in K_{0,1}$. If $(x_k, y_k) \xrightarrow{m} (x, y)$, then $(x, y) \in K_{0,1}$ in view of Lemma 5.2, and $\sigma(x_k) \to \sigma(x)$. From $(x_k, y_k) \to (x, y)$ in $K_{0,0}$ it follows that $\varphi(x_k, y_k) \to \varphi(x, y)$ in F'_0, so that

$$\Phi(x_k, y_k) \to \Phi(x, y) \text{ in } K'_{0,0}.$$

By definition of a strong bundle map, the map $\Phi : K_{0,1} \to K'_{0,1}$ is continuous. From the continuity we conclude $\lim_{k\to\infty} \varphi(x_k, \rho_{x_k} y) = \varphi(x, \rho_x y)$ in $(F')_1$

for all $y \in F_1$. By means of the uniform boundedness principle of linear functional analysis we deduce that the sequence of bounded linear operators

$$T_k(h) := \varphi(x_k, \rho_{x_k}(h))$$

belonging to $\mathcal{L}(F_1, F_1')$, have uniformly bounded operator norms so that $\|T_k\| \leq C$ for all k. Consequently, $\|\varphi(x_k, y_k)\|_1 \leq C \cdot \|y_k\|_1$. From the weak convergence $y_k \rightharpoonup y$ in F_1 we know that $\|y_k\|_1$ is also a bounded sequence. Hence $\|\varphi(x_k, y_k)\|_1$ is a bounded sequence. Because F_1' is a reflexive Banach space, every subsequence of the bounded sequence $\varphi(x_k, y_k)$ in F_1' possesses a subsequence having a weak limit in F_1'. The limit is necessarily equal to $\varphi(x, y)$. Summarizing we have proved that

$$\sigma(x_k) \to \sigma(x) \quad \text{in } O'$$

and

$$\mathrm{pr}_2 \circ \Phi(x_k, y_k) \rightharpoonup \mathrm{pr}_2 \circ \Phi(x, y) \quad \text{in } (F')_1$$

i.e., on level 1.

The discussion shows that the **mixed convergence is invariant under M-polyfold chart transformations** and hence an intrinsic concept for M-polyfold bundles having reflexive fibers so that we can introduce the following definition.

Definition 5.4. *If $p : Y \to X$ is a strong M-polyfold bundle having reflexive fibers, then a sequence $y_k \in Y_{0,1}$ is said to* **converge in the m-sense** *to $y \in Y_{0,1}$, symbolically*

$$y_k \xrightarrow{m} y \quad \text{in } Y_{0,1},$$

if the underlying sequence $x_k = p(y_k) \in X$ converges in X to an element x and if there exists a strong M-polyfold bundle chart Φ around the point $x \in X$ satisfying

$$pr_2 \circ \Phi(x_k, y_k) \rightharpoonup pr_2 \circ \Phi(x, y)$$

weakly in $(F')_1$, i.e., on level 1.

As shown above, the definition does not depend on the choice of the local M-polyfold bundle chart.

For the general perturbation theory we introduce another concept in order to estimate the size of perturbations.

Definition 5.5. *An* **auxiliary norm** N *for the strong M-polyfold bundle* $p : Y \to X$ *consists of a continuous map* $N : Y_{0,1} \to [0, \infty)$ *having the following properties.*

- *For every* $x \in X$*, the induced map* $N|(Y_{0,1})_x \to [0, \infty)$ *on the fiber* $(Y_{0,1})_x$ *is a complete norm.*
- *If* $y_k \xrightarrow{m} y$*, then*
$$N(y) \leq \liminf_{k \to \infty} N(y_k).$$
- *If* $N(y_k)$ *is a bounded sequence and the underlying sequence* x_k *converges to* $x \in X$*, then* y_k *has an m-convergent subsequence.*

Using the paracompactness of X, one can easily construct auxiliary norms as follows.

Proposition 5.6. *Let* $p : Y \to X$ *be a fillable strong M-polyfold bundle having reflexive fibers. Then there exists an auxiliary norm.*

Proof. Construct for every $x \in X$ via local coordinates a norm $N_{U(x)}$ for $Y_{0,1}|U(x)$ where $U(x)$ is a small open neighborhood of $x \in X$. This is defined by $N_{U(x)}(y) = \| pr_2 \circ \Psi(y) \|_1$, where Ψ is a strong M-polybundle chart and pr_2 is the projection onto the fiber part. Observe that m-convergence of y_k to some y in $Y_{0,1}|U(x)$ implies weak convergence of $pr_2 \circ \Psi(y_k)$. Using the convexity of the norm and standard properties of weak convergence we conclude
$$N_{U(x)}(y) \leq \liminf_{k \to \infty} N_{U(x)}(y_k).$$
At this point we have local expressions for auxiliary norms which cover X. Using the paracompactness of X we can find a subordinate partition of unity $(\chi_\lambda)_{\lambda \in \Lambda}$ and define
$$N = \sum \chi_\lambda N_{U(x_\lambda)}.$$
Since the family (χ_λ) is locally finite, the above sum is well-defined. If $N(y_k)$ is bounded and the underlying sequence x_k converges to some $x \in X$ it follows in local coordinates that the representative of y_k is bounded on level 1. In view of the reflexivity, the sequence y_k possesses an m-convergent subsequence. Hence N is an auxiliary norm and the proof of Proposition 5.6 is complete. ■

The following remarks will be important in the proof of the compactness Theorem 5.9. Consider a local strong M-polyfold bundle $K^{\mathcal{R}} \to O$ in which O is an open subset of the splicing core $K^{\mathcal{S}}$ associated with the splicing $\mathcal{S} = (\pi, E, V)$. The splicing core $K^{\mathcal{R}} = \{((v,e),u) \in O \oplus F |\, \rho_{(v,e)}(u) = u\}$ is associated with the strong bundle splicing $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$. Let $N : (K^{\mathcal{R}})_{0,1} \to [0, \infty)$ be an auxiliary norm of the bundle $K^{\mathcal{R}} \to O$. Associated with the complementary splicing $\mathcal{R}^c = (1-\rho, F, (O, \mathcal{S}))$, is the local strong M-poylofd bundle $K^{\mathcal{R}^c} \to O$. In view of Proposition 5.6, the bundle $K^{\mathcal{R}^c} \to O$ can be equipped with an auxiliary norm $N^c : (K^{\mathcal{R}^c})_{0,1} \to [0, \infty)$.

Recalling that $O \triangleleft F$ is the sc-Banach space $O \oplus F$ equipped with the bi-filtration $O_m \oplus F_k$ where $m \geq 0$ and $0 \leq k \leq m+1$, and that $O \triangleleft F = K^{\mathcal{R}} \oplus_O K^{\mathcal{R}^c}$, we define the function $\overline{N} : O_0 \oplus F_1 \to [0, \infty)$ by

$$\overline{N}(x, h) = N(x, \rho_x(h)) + N^c(x, (1 - \rho_x)(h)) \tag{22}$$

where $x \in O_0$ and $h \in F_1$.

Lemma 5.7. *The function $\overline{N} : O_0 \oplus F_1 \to [0, \infty)$ is an auxiliary norm.*

Proof. Clearly, $\overline{N}$ is continuous and for fixed $x \in O_0$, the restriction $\overline{N}(x, \cdot) : F_1 \to [0, \infty)$ defines a complete norm. Assume that $(x_n, h_n) \xrightarrow{m} (x, h)$. Then $x_n \to x$ and $h_n \rightharpoonup h$ in F_1. Since F_1 is compactly embedded in F_0, we have $h_n \to h$ in F_0 and since $\rho : O_0 \oplus F_0 \to F_0$ is continuous, it follows that $\rho_{x_n}(h_n) \to \rho_x(h)$ in F_0. By assumption, $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ is a strong bundle splicing. Hence $\mathcal{R}^1 = (\rho, F^1, (O, \mathcal{S}))$ is a general splicing. This implies implies that $\rho : O_0 \oplus F_1 \to F_1$ is continuous and $\rho_{x_n} : F_1 \to F_1$ is a bounded linear projection for every x_n. Since the sequence (h_n) being weakly convergent is bounded in F_1, it follows from the uniform boundedness principle that the sequence $(\rho_{x_n}(h_n))$ is bounded in F_1. After possibly taking a subsequence we may assume that there is $h' \in F_1$ such that $\rho_{x_n}(h_n) \rightharpoonup h'$ in F_1. Therefore, $\rho_{x_n}(h_n) \to h'$ in F_0 and hence $h' = \rho_x(h)$ and $\rho_x(h) \in F_1$. We conclude that $\rho_{x_n}(h_n) \rightharpoonup \rho_x(h)$ and $(1 - \rho_{x_n})(h_n)\rho \rightharpoonup (1 - \rho_x(h))$ in F_1. Consequently,

$$\begin{aligned}
\overline{N}(x, h) &\leq \overline{N}(x, \rho_x(h)) + \overline{N}(x, (1 - \rho_x)(h)) \\
&= N(x, \rho_x(h)) + N^c(x, (1 - \rho_x)(h)) \\
&\leq \liminf N(x_n, \rho_{x_n}(h_n)) + \liminf N^c(x_n, (1 - \rho_{x_n})(h_n)) \\
&\leq \liminf [N(x_n, \rho_{x_n}(h_n)) + N^c(x_n, (1 - \rho_{x_n})(h_n))] \\
&= \liminf \overline{N}(x_n, h_n)
\end{aligned}$$

as claimed. Finally, assume that $\overline{N}(x_n, h_n)$ is bounded and $x_n \to x$. Then $N(x_n, \rho_{x_n}(h_n))$ and $N^c(x_n, (1-\rho_{x_n})(h_n))$ are bounded and since N and N^c are auxiliary norms, it follows that after possibly taking subsequences $\rho_{x_n}(h_n) \rightharpoonup h'$ and $(1-\rho_{x_n})(h_n) \rightharpoonup h''$ in F_1. Hence $h_n = \rho_{x_n}(h_n) + (1-\rho_{x_n})(h_n) \rightharpoonup h = h' + h''$ in F_1. This completes the proof of the lemma. ■

The function $\overline{N} : O_0 \oplus F_1 \to [0, \infty)$ is continuous and for fixed $x \in O_0$, the restriction $\overline{N}(x, \cdot)$ to the space F_1 is a complete norm on F_1. It follows that there exists a positive constant C_x such that

$$\overline{N}(x, h) \leq C_x \cdot \|h\|_1$$

for all $h \in F_1$. Therefore, by the bounded inverse theorem of linear functional analysis applied to the identity map from the Banach space $(F_1, \|\cdot\|_1)$ to the Banach space $(F_1, \overline{N}(x, \cdot))$, the two norms $\|\cdot\|_1$ and $\overline{N}(x, \cdot)$ are equivalent, i.e., there exist two positive constants $c_x < C_x$ such that

$$c_x \cdot \|h\|_1 \leq \overline{N}(x, h) \leq C_x \cdot \|h\|_1 \tag{23}$$

for all $h \in F_1$. The next lemma claims a local uniform estimates in $x \in O$.

Lemma 5.8. *Let $\overline{N} : O_0 \oplus F_1 \to [0, \infty)$ be an auxiliary norm as defined* (22). *Fix a point $x_0 = (v_0, e_0) \in O_0$. Then there exist two positive constants $c < C$ and an open neighborhood $U \subset O$ of x_0 such that*

$$c\|h\|_1 \leq \overline{N}(x, h) \leq C\|h\|_1$$

for all $x \in U$ and all $h \in F_1$.

Proof. By the continuity of $\overline{N}$ we find an open neighborhood $V_1 \subset O$ of x_0 and a constant $C > 0$ such that $\overline{N}(x, h) \leq C\|h\|_1$ for all $x \in V_1$ and $h \in F_1$. We claim that there exist a positive constant c_0 and an open neighborhood $V_2 \subset O$ of x_0 such that

$$c_0 \cdot \overline{N}(x_0, h) \leq \overline{N}(x, h) \tag{24}$$

for all $x \in V_2$ and $h \in F_1$, so that the lemma follows from (23). In order to prove the estimate (24) we argue by contradiction and assume that there are two sequences $(x_n) \subset O$ and $(h_n) \subset F_1$ satisfying $x_n \to x_0$, $\|h_n\|_1 = 1$, and

$$\frac{1}{n}\overline{N}(x_0, h_n) \geq \overline{N}(x_n, h_n) \quad \text{for } n \geq 0.$$

Set $h_n' = h_n/\overline{N}(x_n, h_n)$. Since $\overline{N}(x_n, h_n') = 1$ and $x_n \to x_0$, it follows from the property (3) of the auxiliary norm that, after possibly passing to a subsequence, the sequence (h_n') converges weakly in F_1. Hence the sequence $\|h_n\|_1$ is bounded and since the norms $\|\cdot\|_1$ and $N(x_0, \cdot)$ are equivalent on F_1, it follows that the sequence $N(x_0, h_n')$ is bounded. Hence $\frac{1}{n}\overline{N}(x_0, h_n') \to 0$ contradicting our assumption $\frac{1}{n}\overline{N}(x_0, h_n') \geq 1$. The proof of the lemma is complete. ■

We recall that the open set $\widehat{O} \subset V \oplus E$ is defined by $\widehat{O} = \{(v, e) \in V \oplus E |\, (v, \pi_v e) \in O\}$. The auxiliary norm $\overline{N} : O \oplus F_1 \to [0, \infty)$ can be extended to $\widehat{O} \oplus F_1$ by defining

$$\widehat{N}((v, e), h) := \overline{N}((v, \pi_v(e)), h) \quad \text{for all } (v, e) \in \widehat{O} \text{ and } h \in F_1. \tag{25}$$

The following result will be useful in compactness investigations.

Theorem 5.9 (Local Compactness). *Consider the fillable strong M-polyfold bundle $p : Y \to X$ having reflexive fibers and let f be a Fredholm section of p. Assume that an auxiliary norm $N : Y_{0,1} \to [0, \infty)$ is given. Then, for every smooth point $q \in X$, there exists an open neighborhood $U(q)$ in X so that the following holds.*

- *The subset $Z \subset X$ defined by*

  $$Z = \{x \in \overline{U(q)}|\; f(x) \in Y_{0,1} \text{ and } N(f(x)) \leq 1\}$$

  is a compact subset of X.

- *Every sequence (x_k) in $\overline{U(q)}$ satisfying $f(x_k) \in Y_{0,1}$ and*

  $$\liminf_{k \to \infty} N(f(x_k)) \leq 1$$

  possesses a convergent subsequence.

Proof. The theorem is of local nature. Hence we shall study in a strong M-polyfold chart the Fredholm section f of the fillable strong local M-polyfold bundle $p : K^{\mathcal{R}} \to O$ where as above O is an open subset of the splicing core $K^{\mathcal{S}}$ and $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ a strong bundle splicing in which $(O, \mathcal{S})$ is the local model of the M-polyfold X. Let $x_0 \in O$ be the smooth point representing $q \in X$ in the local chart. In view of Definition 3.6 of a Fredholm

section, there exists a filled version $\overline{f}:\widehat{O}\to\widehat{O}\triangleleft F$ of the section f of the form

$$\overline{f}(x)=f(x)+f^c(x)$$

for x contained in the open set $\widehat{O}\subset V\oplus E$ defined by $\widehat{O}=\{(v,e)\in V\oplus E|\,(v,\pi_v e)\in O\}$ where f^c is the filler. In the local chart we have the auxiliary norms $N:(K^{\mathcal{R}})_{0,1}\to[0,\infty)$ and $N^c:(K^{\mathcal{R}^c})_{0,1}\to[0,\infty)$. We define by the formula (22) the auxiliary norm $\overline{N}:O\oplus F_1\to[0,\infty)$ and finally obtain the auxiliary norm $\widehat{N}((v,e),h):=\overline{N}((v,\pi_v(e)),h)$ on $\widehat{O}\oplus F_1$. We claim that in order to prove Theorem 5.9 for the section f, it is sufficient to prove the following statement for the filled section $\overline{f}$ of the bundle $\widehat{O}\triangleleft F\to\widehat{O}$.

(∗) There exists an open neighborhood $\widehat{U}(x_0)\subset\widehat{O}$ of x_0 having the property that every sequence $(\widehat{x}_n)\subset\widehat{U}(x_0)$ satisfying $\overline{f}(\widehat{x}_n)\in\widehat{O}\oplus F_1$ and $\liminf\widehat{N}(\overline{f}(x_n))\leq 1$ possesses a convergent subsequence.

Indeed, assume that $(*)$ holds true and define the open neighborhood $U(x_0)=\widehat{U}(x_0)\cap O$. Take a sequence $x_n\in U(x_0)$ satisfying $f(x_n)\in O\oplus F_1$ and $\liminf N(f(x_n))\leq 1$. Since $x_n\in O$, it follows from the definition that the filler term vanishes, $f^c(x_n)=0$, and hence $\widehat{N}(\overline{f}(x_n))=N(f(x_n))+N^c(f^c(x_n))=N(f(x_n))$. Therefore, $\liminf\widehat{N}(\overline{f}(x_n))\leq 1$ so that by $(*)$ the sequence (x_n) has indeed a converging subsequence as claimed.

It remains to prove the statement $(*)$ for the filled section. By Definition 3.6, after a further change of coordinates, the filled section if of the form

$$f:V\oplus W\to\mathbb{R}^N\oplus W$$

where $V\subset\mathbb{R}^n$ is a partial quadrant and f is defined in a sufficiently small closed neighborhood $\overline{U}_0$ of the origin in $(V\oplus W)_0$. Moreover, if $P:\mathbb{R}^N\oplus W\to W$ is the canonical projection, then

$$P[f(a,w)-f(0)]=w-B(a,w).$$

is an sc^0-contraction germ in the sense of Definition 2.1. In addition, we have the transported auxiliary norm $\widehat{N}$ defined on $\overline{U}_0\oplus(\mathbb{R}^N\oplus W)_1$. In view of Lemma 5.8, possibly choosing a smaller neighborhood, we may assume that $\widehat{N}(x,h)=\|h\|_1$ on $(\mathbb{R}^N\oplus W)_1$.

We now consider the set of $(a,w)\in\overline{U}_0$ satisfying $f(a,w)\in(\mathbb{R}^N\oplus W)_1$ and

$$\|f(a,w)\|_1\leq 1.$$

The section f splits according to the splitting of the target space into

$$f(a,w) = \begin{bmatrix} (1-P)f(a,w) \\ Pf(a,w) \end{bmatrix} = \begin{bmatrix} (1-P)f(a,w) \\ w - B(a,w) + Pf(0) \end{bmatrix}.$$

Using the contraction property one finds for given $u \in W$ close to $Pf(0)$ and given $a \in V$ close to 0 a unique $w(a,u) \in W$ solving the equation $Pf(a, w(a,u)) = u$. Moreover, the map $(a,u) \mapsto w(a,u)$ is continuous on the 0-level. Now, given a sequence $(a_k, w_k) \in \overline{U}_0$ such that

$$f(a_k, w_k) =: (b_k, u_k)$$

belongs to $(\mathbb{R}^N \oplus W)_1$ and satisfies

$$\liminf_{k\to\infty} \|f(a_k, w_k)\|_1 \leq 1,$$

we have the equations

$$\begin{aligned} Pf(a_k, w_k) &= u_k \\ (1-P)f(a_k, w_k) &= b_k \\ w_k &= w(a_k, u_k). \end{aligned}$$

We shall show that (a_k, w_k) possesses a convergent subsequence in $\overline{U}_0$. By assumption, $(\mathbb{R}^N \oplus W)_1$ is a reflexive Banach space so that going over to a subsequence and using the compact embedding $W_1 \subset W_0$,

$$\begin{aligned} (b_k, u_k) &\rightharpoonup (b', u') \quad \text{in } (\mathbb{R}^N \oplus W)_1 \\ (b_k, u_k) &\to (b', u') \quad \text{in } (\mathbb{R}^N \oplus W)_0, \end{aligned}$$

and $\|(b', u')\|_1 \leq 1$. In $\mathbb{R}^N$ we may assume $a_k \to a' \in V$. Consequently, $w_k = w(a_k, u_k) \to w' := w(a', u')$. Hence $f(a', w') = (b', u')$ and $\|f(a', w')\|_1 \leq 1$. The proof of Theorem 5.9 is complete. ■

5.2 Proper Fredholm Sections

In this section we introduce the important class of proper Fredholm sections.

Definition 5.10. *A Fredholm section f of the fillable strong M-polyfold bundle $p : Y \to X$ is called* **proper** *provided $f^{-1}(0)$ is compact in X.*

The first observation is the following.

Theorem 5.11 (∞-Properness)**.** *Assume that f is a proper Fredholm section of the fillable strong M-polyfold bundle $p : Y \to X$. Then $f^{-1}(0)$ is compact in X_∞.*

Proof. Properness implies by definition that $f^{-1}(0)$ is compact on level 0. Of course, $f^{-1}(0)$ is a subset of X_∞ since f is regularizing. Assume that (x_k) is a sequence of solutions of $f(x) = 0$. We have to show that it possesses a subsequence converging to some solution x_0 in X_∞. After taking a subsequence we may assume that $x_k \to x_0$ on level 0. We choose a contraction germ representation for $[f, x_0]$. After a suitable change of coordinates the sequence $((a_k, w_k))$ corresponds to (x_k), the point $(0, 0)$ to x_0, and

$$w_k = B(a_k, w_k),$$

with $(a_k, w_k) \to (0, 0)$ on level 0. Consequently,

$$w_k = \delta(a_k)$$

for the map $a \to \delta(a)$ constructed using Banach's fixed point theorem on level 0. We know that for every level m there is an open neighborhood O_m of 0 so that $\delta_m : O_m \to W_m$ is continuous given by Banach's fixed point theorem on the m-level. We may assume that $O_{m+1} \subset O_m$. Fix a level m. For k large enough, we have $a_k \in O_m$. Then $\delta_m(a_k)$ is the same as $\delta(a_k) = w_k$. Hence $a_k \to 0$ implies $w_k \to 0$ on level m. Consequently, $(a_k, w_k) \to (0, 0)$ on every level, implying convergence on the ∞-level. Hence $x_k \to x_0$ in X_∞ as claimed. ■

As a consequence of the local Theorem 5.9 we obtain the following global result for proper Fredholm sections.

Theorem 5.12. *Let $p : Y \to X$ be a fillable strong M-polyfold bundle with reflexive fibers and assume that f is a proper Fredholm section. Assume that N is a given auxiliary norm. Then there exists an open neighborhood U of the compact set $S = f^{-1}(0)$ so that the following holds true.*

- *For every section $s \in \Gamma^+(p)$ having its support in U and satisfying $N(s(x)) \leq 1$, the section $f + s$ is a proper Fredholm section of the fillable strong M-polyfold bundle $p^1 : Y^1 \to X^1$ and $(f + s)^{-1}(0) \subset U$.*

- *Every sequence (x_k) in $\overline{U}$ satisfying $f(x_k) \in Y_{0,1}$ and*

$$\liminf_{k \to \infty} N(f(x_k)) \leq 1$$

  possesses a convergent subsequence.

Proof. We know from the local Fredholm theory (Theorem 3.9) that $f+s$ is a Fredholm section of the fillable bundle $p^1 : Y^1 \to X^1$ if s is an sc$^+$-section. For every $q \in S = f^{-1}(0)$ there exists an open neighborhood $U(q)$ having the properties as described in Theorem 5.9. Since S is a compact set we find finitely many q_i so that the open sets $U(q_i)$ cover S. We denote their union by U. Next assume that the support of $s \in \Gamma^+(p)$ is contained in U. If $(f+s)(x) = 0$, then necessarily $x \in U$ because otherwise $s(x) = 0$ implying $f(x) = 0$ and hence $x \in S \subset U$, a contradiction. Consequently, the set of solutions of $(f+s)(x) = 0$ belongs to $\{x \in U |\ f(x) \in Y_{0,1} \text{ and } N(f(x)) \leq 1\}$ which by construction and Theorem 5.9 is contained in a finite union of compact sets. The proof of Theorem 5.12 is complete. ∎

5.3 Transversality and Solution Set

From our local considerations in the previous chapter we shall deduce the main results about the solution set of Fredholm sections.

Definition 5.13. *A Fredholm section f of the fillable strong M-polyfold bundle $p : Y \to X$ is said to be* **transversal** *to the zero section if at every solution x of $f(x) = 0$ the linearization $f'(x) : T_xX \to Y_x$ is surjective.*

The first result is concerned with a proper Fredholm section of a fillable strong M-polyfold bundle $p : Y \to X$ where X has no boundary.

Theorem 5.14. *Let f be a proper and transversal Fredholm section of the fillable strong M-polyfold bundle $p : Y \to X$ where X has no boundary. Then the solution set $\mathcal{M} = f^{-1}(0)$ is a smooth compact manifold (without boundary).*

Proof. Near a solution x of $f(x) = 0$ we take suitable strong local bundle coordinates $\Phi : Y|U(x) \to K$ covering an sc-diffeomorphism $\varphi : U(x) \to O$ satisfying $\varphi(x) = 0$. Here $K \to O$ is the fillable local strong M-polyfold bundle. The local expression f_Φ of the Fredholm section f is also a Fredholm section, has 0 as a solution, and its linearization $f'_\Phi(0)$ is, by assumption, a

surjective sc-Fredholm operator. Hence, by Theorem 4.6, there exists a good parametrization of the solution set near 0,

$$\Gamma_\Phi(n) = n + A(n)$$

defined on an open neighborhood of 0 in the kernel of $f'_\Phi(0)$. Then we define on a small open neighborhood of $0 \in \ker f'(0)$ the map $\Gamma_x = \varphi^{-1} \circ \Gamma_\Phi \circ T\varphi(x)$. We can carry out this construction near every solution x of $f(x) = 0$. By the calculus of good parameterizations in section 4.1, the transition maps $\Gamma_x^{-1} \circ \Gamma_y$ are sc-smooth and since they are defined and take values in smooth finite dimensional vector spaces they are of class C^∞. Consequently, the inverses of the maps Γ_x define a smooth atlas for $\mathcal{M}$.

■

In the case where the ambient M-polyfold has a boundary, the solution set of a proper and transversal Fredholm section f of the bundle $p : Y \to X$ is in general not a smooth compact manifold with boundary due to a possible bad position of the solution set to the boundary. In order to draw conclusions about the nature of the solution set we need some knowledge about the position of the kernels of the linearized operators at solutions on the boundary. Some of these positions are in some sense generic and genericity questions are being investigated in the next section. We continue with a definition.

Definition 5.15. *Let f be a Fredholm section of the fillable strong M-polyfold bundle $p : Y \to X$, where X possibly has a boundary. The section f is called in* **good position** *if at every point $x \in f^{-1}(0)$ the Fredholm section f is in good position to the corner structure of X as defined in Definition 4.16. Recall that this requires, in particular, that the linearisation $f'(x)$ is surjective at every solution x of $f(x) = 0$.*

The next result gives a generalization of the previous theorem. We consider the situation of a proper Fredholm section on a M-polyfold with possible boundary.

Theorem 5.16. *Consider a proper Fredholm section f of the fillable strong M-polyfold bundle $p : Y \to X$ which is in good position to the corner structure. Then the solution set $f^{-1}(0)$ carries in a natural way the structure of a smooth compact manifold with boundary $\partial f^{-1}(0)$ with corners where $\partial f^{-1}(0) = f^{-1}(0) \cap \partial X$.*

The proof of the theorem is almost identical to the proof of Theorem 5.14 with the exception that, in addition, the local expression f_Φ is in good position to the corner structure of O at 0 and we use Theorem 4.18 to guarantee the existence of the parametrization Γ_Φ in this case.

Proof. Near a solution x of $f(x) = 0$ we take suitable strong local bundle coordinates,

$$\Phi : Y|U(x) \to K$$

covering an sc-diffeomorphism $\varphi : U(x) \to O$ satisfying $\varphi(x) = 0$, where $K \to O$ is the local strong M-polyfold bundle. The local expression f_Φ of the section f has 0 as a solution and f_Φ, by assumption, is in good position to the corner structure of O at the point 0. Hence, by Theorem 4.18, there is a good parametrization of the solution set near 0,

$$\Gamma_\Phi(n) = n + A(n).$$

This parametrization is defined on a relatively open neighborhood of 0 in a partial quadrant $P := \ker f'_\Phi(0) \cap C$ of the kernel $\ker f'_\Phi(0)$. We define the map $\Gamma_x = \varphi^{-1} \circ \Gamma_\Phi \circ T\varphi(x)$ on a small relatively open neighborhood of 0 in a partial quadrant contained in $\ker(f'(x)) \subset T_xX$. We can carry out this construction near every solution $x \in X$ of $f(x) = 0$. By our discussion of the calculus of good parameterizations the transition maps are smooth (i.e., of class C^∞). Hence the inverses of the maps Γ_x define a smooth atlas equipping the solution set $f^{-1}(0)$ with the structure of a smooth compact manifold with boundary with corners. ∎

If we have a proper Fredholm section of a fillable strong M-polyfold bundle we can introduce the notion of being in general position. Intuitively this means two things. First of all it requires that at a solution the linearisation is surjective. Secondly the solution set should avoid finite intersections of (local) faces if the dimension of this intersection has too large codimension. For example, a one-dimensional solution set would only hit good boundary points ($d(x) = 1$) and would not hit corners ($d(x) > 1$) and similarly a two-dimensional solution set would not hit boundary points x with $d(x) \geq 3$. The importance of the notion of being in general position comes from the fact that it can be achieved by an arbitrarily small perturbation as shown in section 5.4. We would, however, like to point out that in SFT sometimes certain symmetry requirements do not allow to bring a solution set into general position by a perturbation compatible with the symmetries. Nevertheless

one can still achieve a good position. This will be studied in [16] and in an entirely abstract framework in [17].

Consider a proper Fredholm section f of a fillable strong M-polyfold bundle $p : Y \to X$ where the M-polyfold X possibly has a boundary. If x is a point in X it belongs to $d(x)$-many local faces. Fixing a smooth point x, we consider the $d(x)$ local faces numbered as $\mathcal{F}^1, \ldots, \mathcal{F}^d$ where $d = d(x)$. Each local face $\mathcal{F}^j$ has the tangent space $T_x\mathcal{F}^j$ at the point $x \in X$. We denote by $T_x^\partial X$ the intersection

$$T_x^\partial X = \bigcap_{1\leq j\leq d} T_x\mathcal{F}^j.$$

If $d(x) = 0$, i.e., if x is an interior point, we define $T_x^\partial X = T_x X$.

Definition 5.17. *The Fredholm section f is in* **general position** *to the boundary ∂X provided the following holds for every solution x of $f(x) = 0$.*

(i) The linearization $f'(x)$ is surjective.

(ii) The kernel of $f'(x)$ is transversal to $T_x^\partial X$ in the tangent space $T_x X$.

Theorem 5.18. *Assume that the proper Fredholm section f of the fillable strong M-polyfold bundle $p : Y \to X$ is in general position to the boundary ∂X. Then the solution set $f^{-1}(0)$ is a smooth compact manifold with boundary with corners.*

This theorem is an immediate consequence of Theorem 5.16 and the following observation.

Lemma 5.19. *If the Fredholm section f of the fillable strong M-polyfold bundle $p : Y \to X$ is in general position, then it is in good position as well.*

Proof. Take a solution x of $f(x) = 0$. The linearization $f'(x)$ is surjective since f is in general position. Now we assume that $d(x) \geq 1$. By assumption, the kernel of $f'(x)$ is transversal to $T_x^\partial X$ implying $\dim \ker f'(x) \geq \text{codim } \mathrm{T}_x^\partial X$ and hence,

$$\dim\ \ker f'(x) \geq d(x).$$

Going into local coordinates, we may assume that the section f is already filled, that $x = 0$ and that $0 \in C = [0, \infty)^d \oplus \mathbb{R}^{n-d} \oplus W$ where $d = d(x)$. The linearisation

$$f'(0) : \mathbb{R}^n \oplus W \to \mathbb{R}^N \oplus W$$

is surjective and its index satisfies $i(f,0) = \dim\ker f'(0) = n - N$ in view of Proposition 3.8. Further, $\ker f'(0)$ is transversal to $\{0\}^d \oplus \mathbb{R}^{n-d} \oplus W$. This implies that $\ker f'(0)$ has an sc-complement $N^\perp$ contained in C. Therefore, $\ker f'(0)$ is neat with respect to C and hence, by Proposition 4.11, in good position to the partial quadrant C. ■

5.4 Perturbations

Using the results from the previous section we shall prove some useful perturbation results. In order to have an sc-smooth partition of unity on the M-polyfold X guaranteed, we shall assume for simplicity throughout this section 5.4 that the **local models of X are splicing cores built on separable sc-Hilbert spaces.** For the sc-smooth partitions of unity on M-polyfolds we refer to section 4.3 in [14].

We begin with a useful lemma.

Lemma 5.20. *Consider the strong M-polyfold bundle $p: Y \to X$ equipped with an auxiliary norm $N: Y_{0,1} \to [0,\infty)$. Assume that the local models of X are built on separable sc-Hilbert spaces. Let x_0 be a smooth point of X, h_0 a smooth point belonging to the fiber Y_{x_0}, and $U \subset X$ an open neighborhood of x_0. Then there exists an sc$^+$-section s of the strong bundle p satisfying*

$$s(x_0) = h_0$$

and having its support in U. Moreover, if $N(h_0) < \varepsilon$ we can choose the section s in such a way that

$$N(s(y)) < \varepsilon.$$

for all $y \in X$.

Proof. The result is local. Hence we take a strong M-polyfold bundle chart $\Phi: p^{-1}(U) \to K^{\mathcal{R}}$ covering the sc-diffeomorphism $\varphi: U \to O$ as defined in Definition 4.8 of [12]. The set O is an open subset of of the splicing core $K^{\mathcal{S}} = \{(v,e) \in V \oplus E | \pi_v(e) = e\}$ associated with the splicing $\mathcal{S} = (\pi, E, V)$ and $K^{\mathcal{R}} = \{((v,e),u) \in O \oplus F | \rho_{(v,e)}(u) = u\}$ is the splicing core associated with the strong bundle splicing $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$. In these coordinates the smooth point $(v_0, e_0) = \varphi(x_0) \in O$ corresponds to x_0. Moreover, $\Phi(x_0, h_0) =$

$((v_0, e_0), h_0') \in K^{\mathcal{R}}$ where h_0' is a smooth point in F, that is, $h_0' \in F_\infty$. For points $(v, e) \in O$ close to (v_0, e_0) we define the local section $s : K^{\mathcal{R}} \to O$ by

$$s(v, e) = ((v, e), \rho_{(v,e)}(h_0')) \in K^{\mathcal{R}}.$$

At the point (v_0, e_0) we then have $\rho_{(v_0,e_0)}(h_0') = h_0'$ and hence $s(v_0, e_0) = ((v_0, e_0), h_0')$ as desired. In view of the definition of a strong bundle splicing, $\rho_{(v,e)}(u) \in F_{m+1}$ if $(v, e) \in O_m \oplus F_m$ and $u \in F_{m+1}$. Moreover, by Definition 4.2 in [12], the triple $\mathcal{R}^1 = (\rho, F^1, (O, \mathcal{S}))$ is also a general sc-splicing which together with the fact that h_0' is a smooth point in F implies that the section s of the bundle $K^{\mathcal{R}^1} \to O$ is sc-smooth. Consequently, s is an sc$^+$-section of the local strong M-polyfold bundle $K^{\mathcal{R}} \to O$. We transport this section by means of the map Φ to obtain a local sc$^+$ -section s of the bundle $p : Y \to X$. It satisfies $s(x_0) = h_0$. Using Lemma 5.7 from [14] we find an sc-smooth bump-function β having its support in U and equal to 1 near x_0 so that the section $\beta \cdot s$ has the desired properties. ■

In the following we shall use the notation

$$\mathcal{M}^f = \{x \in X |\, f(x) = 0\}$$

for the solution set of the section f of the strong M-polyfold bundle $p : Y \to X$.

Theorem 5.21. *Assume that f is a Fredholm section of the fillable strong M-polyfold bundle $p : Y \to X$ and assume $\partial X = \emptyset$. We assume that the sc-structure on X is is built on separable sc-Hilbert spaces. Let N be an auxiliary norm. Assume that U is the open neighborhood of $f^{-1}(0)$ guaranteed by Theorem 5.12 having the property that for every section $s \in \Gamma^+(p)$ whose support lies in U and which satisfies $N(s(x)) \leq 1$, the section $f + s$ is a proper Fredholm section of the fillable strong M-polyfold bundle $p^1 : Y^1 \to X^1$. Denote the space of all such sections in $\Gamma^+(p)$ as just described by $\mathcal{O}$. Then given $t \in \frac{1}{2}\mathcal{O}$ and $\varepsilon \in (0, \frac{1}{2})$, there exists a section $s \in \mathcal{O}$ satisfying $s - t \in \varepsilon\mathcal{O}$ so that $f + s$ has at every solution x of $(f + s)(x) = 0$ a linearization which is surjective. In particular, the solution set*

$$\mathcal{M}^{f+s} = \{x \in X^1 |\, (f + s)(x) = 0\}$$

is a smooth compact manifold without boundary in view of Theorem 5.14.

Proof. If $t \in \frac{1}{2}\mathcal{O}$, then, in view of Theorem 5.12, $f + t$ is a proper Fredholm section of the bundle $p^1 : Y^1 \to X^1$. The solution set $S = (f + t)^{-1}(0)$ is compact in X_∞ in view of Theorem 5.11 and at every solution $x \in S$ the linearization $(f + t)'(x)$ is sc-Fredholm by Proposition 3.8. Fixing $x_0 \in S$, there is an sc-splitting of the image space of the linearization $(f + t)'(x_0)$ into the range and a smooth finite dimensional cokernel in which we choose the basis $e_1, \ldots, e_m$. It consists of smooth vectors. By Lemma 5.20, there exist finitely many sections $\widetilde{s}_1, \ldots, \widetilde{s}_m \in \Gamma^+(p)$ having their supports in U and satisfying $\widetilde{s}_1(x_0) = e_1, \ldots, \widetilde{s}_m(x_0) = e_m$. Therefore, the linear map

$$(\lambda_1, \ldots, \lambda_m, h) \mapsto (f + t)'(x_0)h + \sum_{j=1}^{m} \lambda_j e_j$$

is surjective. Now, we view Y as a fillable strong M-polyfold bundle over $\mathbb{R}^m \oplus X$. Then the mapping $\widetilde{F} : \mathbb{R}^m \oplus X^1 \to Y^1$, defined by

$$\widetilde{F}(\lambda, x) = (f + t)(x) + \sum_{j=1}^{m} \lambda_j \widetilde{s}_j,$$

is a Fredholm section of the fillable bundle $Y^1 \to \mathbb{R}^m \oplus X^1$ in view of Theorem 3.9. It has the property that $\widetilde{F}(0, x_0) = 0$ and that the linearization $\widetilde{F}'(0, x_0)$ is surjective. Using the good parametrizations guaranteed by Theorem 4.6, we find an open neighborhood $V((0, x_0))$ of the point $(0, x_0) \in \{0\} \oplus S$ on which the linearization $\widetilde{F}'(\lambda, x)$ is surjective at points (λ, x) in the solution set $\{\widetilde{F}(\lambda, x) = 0\}$. Since $\{0\} \oplus S$ is compact, we can cover $\{0\} \oplus S$ with finitely many such open neighborhoods $V((0, x_i))$ for $i = 1, \ldots, N$. Adding up all the sections constructed in these open sets, we obtain the finitely many sections $s_1, \ldots, s_k \in \Gamma^+(p)$ all having their supports in U so that the Fredholm section $F : \mathbb{R}^k \oplus X^1 \to Y^1$ defined by

$$F(\lambda, x) = (f + t)(x) + \sum_{j=1}^{k} \lambda_j s_j(x)$$

has the following property. The linearization $F'(\lambda, x)$ is surjective at points (λ, x) contained in an open neighborhood of $\{0\} \oplus S$ and solving the equation $F(\lambda, x) = 0$. For $\delta > 0$ sufficiently small we take the open ball $B_\delta(0) \subset \mathbb{R}^k$ centered at the origin and of radius δ. Then the solution set

$$S_\delta := \{(\lambda, x) \in B_\delta(0) \oplus X^1 \,|\, F(\lambda, x) = 0\}$$

is a smooth finite dimensional manifold using the good parametrizations guaranteed by Theorem 4.6 and arguing as in the proof of Theorem 5.14. We define the smooth map $\beta : S_\delta \to \mathbb{R}^k$ as the composition $\beta = \pi \circ j$,

$$S_\delta \xrightarrow{j} \mathbb{R}^k \oplus X \xrightarrow{\pi} \mathbb{R}^k$$

where j is the injection map and π is the projection $\pi(\lambda, x) = \lambda$. By Sard's theorem, we find a regular value $\lambda^* \in \mathbb{R}^k$ of the map β arbitrary close to 0. In view of the surjectivity of $F'(\lambda, x)$, the map $x \mapsto F(\lambda^*, x)$ has therefore the property that the linearization $D_2F(\lambda^*, x)$ is surjective at all solutions $x \in X$ of the equation $F(\lambda^*, x) = 0$. Moreover, the map $F(\lambda^*, \cdot)$ is a proper Fredholm section of the bundle $p^1 : Y^1 \to X^1$ in view of Theorem 5.12. Hence, by Theorem 5.14, the solution set $\{x \in X^1 | F(\lambda^*, x) = 0\}$ is a smooth compact manifold without boundary and the proof of Theorem 5.21 is complete. ■

Our main perturbation result concerning Fredholm problems on M-polyfolds possessing boundaries is as follows.

Theorem 5.22. *Let $p : Y \to X$ be a fillable strong M-polyfold bundle having nonempty boundary ∂X and assume that $f : X \to Y$ is a proper Fredholm section. We assume that the sc-structure of X is built on separable sc-Hilbert spaces. Fix an auxiliary norm N and assume that $U \subset X$ is an open neighborhood of the compact solution set $\mathcal{M}^f$ guaranteed by Theorem 5.12. Denote by $\mathcal{O}$ the set of sections*

$$\mathcal{O} = \{s \in \Gamma^+(p) \mid \operatorname{supp}(s) \subset U \text{ and } N(s(x)) \leq 1 \text{ for all } x \in U\}.$$

Then, given $t \in \frac{1}{2}\mathcal{O}$ and $\varepsilon \in (0, \frac{1}{2})$, there exists a section $s \in \mathcal{O}$ satisfying $s - t \in \varepsilon\mathcal{O}$ so that the solution set

$$\mathcal{M}^{f+s} = \{x \in X^1 | \, (f + s)(x) = 0\}$$

is in general position to the boundary ∂X. In particular, $\mathcal{M}^{f+s}$ is a smooth compact manifold with boundary with corners in view of Theorem 5.18.

Proof. Take $t \in \frac{1}{2}\mathcal{O}$. Then the solution set $\mathcal{M}^{f+t}$ is compact in X as well as in X_∞ by Theorem 5.11. If x is a smooth point in X, we denote by $\mathcal{F}^1, \ldots, \mathcal{F}^d$ the local faces of X at the point x. Here $d = d(x)$. Each local

face $\mathcal{F}^j$ has the tangent space $T_x\mathcal{F}^j$ at the point $x \in X$. Recall that $T_x^\partial X$ is the intersection $T_x^\partial X = \bigcap_{1\leq j\leq d} T_x\mathcal{F}^j$. If $d(x) = 0$, i.e., if x is an interior point, we set $T_x^\partial X = T_xX$. We shall proceed in several steps starting with the following lemma.

Lemma 5.23. *There exist finitely many sc$^+$-sections $s_j \in \Gamma^+(p)$ for $1 \leq j \leq l$, supported in U so that the map $F : \mathbb{R}^l \oplus X^1 \to Y^1$ defined by*

$$F(\lambda, x) := (f + t)(x) + \sum_{j=1}^{l} \lambda_j s_j(x)$$

has the following properties. There exists a small $\varepsilon > 0$ so that the solution set

$$\overline{S}_\varepsilon = \{(\lambda, x) \in \mathbb{R}^l \oplus X^1 |\, F(x, \lambda) = 0 \text{ and } |\lambda| \leq \varepsilon\}$$

is compact in $\mathbb{R}^l \oplus X_\infty$ and contained in $\overline{B}_\varepsilon \oplus U$. Moreover, at every solution $(\lambda, x) \in \overline{S}_\varepsilon$, the section F has the following properties.

- *(i) the linearization $F'(\lambda, x)$ is surjective,*
- *(ii) the kernel of $F'(\lambda, x)$ is transversal to the subspace $T^\partial_{(\lambda,x)}(\mathbb{R}^l \oplus X)$ of the tangent space $T_{(\lambda,x)}(\mathbb{R}^l \oplus X)$,*
- *(iii) for every subset $\sigma \subset \{1, \ldots, d(x)\}$, where $d(x)$ is the order of the degeneracy of the point x, the lineraization $F'(\lambda, x)$ restricted to the tangent space $T_{(\lambda,x)}(\mathbb{R}^l \oplus \bigcap_{j\in\sigma} \mathcal{F}^j)$ is surjective and the kernel of this restriction is transversal to the subspace $T^\partial_{(\lambda,x)}(\mathbb{R}^l \oplus \bigcap_{j\in\sigma} \mathcal{F}^j)$ in the tangent space $T_{(\lambda,x)}(\mathbb{R}^l \oplus \bigcap_{j\in\sigma} \mathcal{F}^j)$.*

Note that $T^\partial_{(\lambda,x)}(\mathbb{R}^l \oplus X) = \mathbb{R}^l \oplus T^\partial_x X$ and $T_{(\lambda,x)}(\mathbb{R}^l \oplus \bigcap_{j\in\sigma} \mathcal{F}^j) = \mathbb{R}^l \oplus T_x \bigcap_{j\in\sigma} \mathcal{F}^j$ where $\sigma \subset \{1, \ldots, d(x)\}$. The codimension of $T^\partial_{(\lambda,x)}(\mathbb{R}^l \oplus X)$ in the tangent space $T_{(\lambda,x)}(\mathbb{R}^l \oplus X)$ is independent of l and is equal to $d(x)$.

Proof of Lemma 5.23. Choose a solution x_0 of $(f + t)(x_0) = 0$. The linearization $(f + t)'(x_0)$ is sc-Fredholm by Proposition 3.8. Hence, in view of Lemma 5.20, we find finitely many sections $s_1, \ldots, s_k \in \Gamma^+(p)$ satisfying $N(s_j(x)) < 1$ for all $x \in X$ and having their supports in U so that the vectors $s_1(x_0), \ldots, s_k(x_0)$ span the smooth sc-complement of the range of the

linearization $(f+t)'(x_0)$. Taking additional sections $s_{k+1}, \ldots, s_m$ in $\Gamma^+(p)$ having their supports in U, we arrange that also the kernel of the linear map

$$(\lambda, h) \mapsto (f+t)'(x_0) \cdot h + \sum_{j=1}^{m} \lambda_j s_j(x_0)$$

is transversal to $\mathbb{R}^m \oplus T^{\partial}_{x_0} X$ in the tangent space $\mathbb{R}^m \oplus T_{x_0} X$. This is done as follows. We first observe that the kernel of the above linear map is equal to $\{0\} \oplus \ker(f+t)'(x_0)$ together with the set of points (λ, h) solving the equation $(f+t)'(x_0)h = -\sum_{j=k+1}^{m} \lambda_j s_j(x_0)$. Since $T^{\partial}_{x_0} X$ has finite codimension in the tangent space $T_{x_0} X$, there are finitely many smooth vectors $h_{k+1}, \ldots, h_m$ such that $\operatorname{span}\{h_{k+1}, \ldots, h_m\} \oplus T^{\partial}_{x_0} X = T_{x_0} X$, using Lemma 2.12 in [12]. Consequently, choosing the additional sections s_j for $k+1 \leq j \leq m$ such that $s_j(x_0) = -(f+t)'(x_0)h_j$, the transversality of the kernel follows. Moreover, multiplying the vectors h_j by suitable small constants we may achieve, in view of Lemma 5.20, that $N(s_j(x)) < 1$ for all $x \in X$ and $k+1 \leq j \leq m$.

Viewing Y as a fillable strong M-polyfold bundle over $\mathbb{R}^m \oplus X$, the map $F : \mathbb{R}^m \oplus X^1 \to Y^1$, defined by

$$F(\lambda, x) = (f+t)(x) + \sum_{k=1}^{m} \lambda_j s_j(x) \tag{26}$$

is a Fredholm section of the bundle $Y^1 \to \mathbb{R}^m \oplus X^1$ having the property that $F(0, x_0) = 0$, that the linearization $F'(0, x_0)$ is surjective, and that the kernel of $F'(0, x_0)$ is transversal to $\mathbb{R}^m \oplus T^{\partial}_{x_0} X$ in $T_{(0,x_0)}(\mathbb{R}^m \oplus X)$. Note that after adding r more sections the kernel of the new linearized operator is automatically transversal to $T^{\partial}_{(0,x_0)}(\mathbb{R}^{m+r} \oplus X)$ in the tangent space $T_{(0,x_0)}(\mathbb{R}^{m+r} \oplus X)$, moreover, the linearized operator is still surjective.

Claim (a) There is an open neighborhood $U(0, x_0)$ of the point $(0, x_0)$ in $\mathbb{R}^m \oplus X$ such that at every solution (λ, x) of $F(\lambda, x) = 0$ contained in $U(0, x_0)$, the linearization $F'(\lambda, x)$ is surjective and the kernel of $F'(\lambda, x)$ is transversal to $T^{\partial}_{(\lambda,x)}(\mathbb{R}^m \oplus X)$ in $T_{(\lambda,x)}(\mathbb{R}^m \oplus X)$.

In order to prove this claim we work in our familiar local coordinates and assume that the section F is already filled and has the normal form

$$F : \mathbb{R}^m \oplus [0, \infty)^d \oplus \mathbb{R}^{n-d} \oplus W \to \mathbb{R}^N \oplus W.$$

The integer d is equal to the order $d = d(x_0)$ of x_0. In these coordinates the point x_0 corresponds to the point $0 \in [0, \infty)^d \oplus \mathbb{R}^{n-d} \oplus W$ so that $(0, x_0)$

corresponds to $(0,0) \in \mathbb{R}^m \oplus [0,\infty)^d \oplus \mathbb{R}^{n-d} \oplus W$. In order to simplify the notation we abbreviate, in abuse of the notation, $X = [0,\infty)^d \oplus \mathbb{R}^{n-d} \oplus W$. Then

$$T^{\partial}_{(0,0)}(\mathbb{R}^m \oplus X) = \mathbb{R}^m \oplus T^{\partial}_0 X = \mathbb{R}^m \oplus \{0\}^d \oplus \mathbb{R}^{n-d} \oplus W$$

and

$$T_{(0,0)}(\mathbb{R}^m \oplus X) = \mathbb{R}^m \oplus \mathbb{R}^d \oplus \mathbb{R}^{n-d} \oplus W.$$

By construction, the kernel N of the linearization $F'(0,0)$ is transversal to the tangent space $T^{\partial}_{(0,0)}(\mathbb{R}^m \oplus X)$. Therefore, there is an sc-complement $N^{\perp}$ of N satisfying $N^{\perp} \subset \mathbb{R}^m \oplus \{0\}^d \oplus \mathbb{R}^{n-d} \oplus W$. In particular, the kernel N is neat with respect to the partial cone $C = \mathbb{R}^m \oplus [0,\infty)^d \oplus \mathbb{R}^{n-d} \oplus W$ according to Definition 4.10. In view of Theorem 4.18, there is a good parametrization of the solution set $\{F = 0\}$ near the solution $(0,0)$. Accordingly there exist an open neighborhood Q of $(0,0)$ in $N \cap C$ and an sc-smooth map

$$A : Q \to N^{\perp} \tag{27}$$

satisfying $A(0,0) = 0$ and $DA(0,0) = 0$ so that mapping $\Gamma(n) = n + A(n)$ parametrizes all the solutions of $F = 0$ near $(0,0)$. Moreover, the linearization $F'(\Gamma(n))$ is surjective for all $n \in Q$. If $\Gamma(n) = (\lambda, x)$ is such a solution, we write $x = (x_1, \ldots, x_d, x') \in [0,\infty)^d \oplus (\mathbb{R}^{n-d} \oplus W)$. Denote by Σ the subset of indices $i \in \{0, \ldots, d\}$ for which $x_i = 0$ and introduce the subspace $\mathbb{R}^{\Sigma} \subset \mathbb{R}^d$ consisting of points $z \in \mathbb{R}^d$ for which $z_i = 0$ if $i \in \Sigma$. Then $T^{\partial}_{(\lambda,x)}(\mathbb{R}^m \oplus X) = \mathbb{R}^m \oplus \mathbb{R}^{\Sigma} \oplus \mathbb{R}^{n-d} \oplus W$ and we have to show that the kernel N' of the linearization $F'(\lambda, x)$ is transversal to $\mathbb{R}^m \oplus \mathbb{R}^{\Sigma} \oplus \mathbb{R}^{n-d} \oplus W$. By the properties of the good parametrization Γ in Definition 4.17, the kernel N' has the representation $N' = \{\delta n + DA(n)\delta n \mid \delta n \in N\}$. If $v \in \mathbb{R}^m \oplus \mathbb{R}^d \oplus \mathbb{R}^{n-d} \oplus W$, then $v = u + u'$ for some $u \in N$ and some $u' \in \mathbb{R}^m \oplus \{0\}^d \oplus \mathbb{R}^{n-d} \oplus W$ because the kernel N and the subspace $\mathbb{R}^m \oplus \{0\}^d \oplus \mathbb{R}^q \oplus W$ are transversal. Now observe that

$$v = (u + DA(n)u) + (u' - DA(n)u)$$

where $u + DA(n)u \in N'$. Denoting by $p : \mathbb{R}^m \oplus \mathbb{R}^d \oplus \mathbb{R}^q \oplus W \to \mathbb{R}^d$ the natural projection, we have $p(u' - DA(n)u) = 0$ because $p(u') = 0$ and $DA(n)u \in N^{\perp} \subset R^m \oplus \{0\}^d \oplus \mathbb{R}^q \oplus W$ so that $p(DA(n)u) = 0$. In particular, the kernel N' is transversal to $T^{\partial}_{(\lambda,x)}(\mathbb{R}^m \oplus X)$ as claimed in (a).

Claim (b) For every solution $(\lambda, x) \in U(0, x_0)$ and every subset σ of $\{1, \ldots, d(x)\}$, the linearization $F'(\lambda, x)$ restricted to $T_{(\lambda,x)}(\mathbb{R}^m \oplus \bigcap_{j\in\sigma} \mathcal{F}^j)$ is surjective and the kernel of of this map is transversal to $T^{\partial}_{(\lambda,x)}(\mathbb{R}^m \oplus \bigcap_{j\in\sigma} \mathcal{F}^j)$ in $T_{(\lambda,x)}(\mathbb{R}^m \oplus \bigcap_{j\in\sigma} \mathcal{F}^j)$.

In order to prove the claim (b) we observe that the point (λ, x) belongs to $d(x)$- many faces $\mathbb{R}^m \oplus \mathcal{F}^j$. The faces $\mathbb{R}^m \oplus \mathcal{F}^j$ are in the coordinates represented as follows. The set $\Sigma = \{i \in \{1, \ldots, d\} |\, x_i = 0\}$ has exactly $d(x)$ elements. For $j \in \Sigma$, the face $\mathcal{F}^j$ is the set

$$\mathcal{F}^j = [0, \infty)^{(j)} \oplus \mathbb{R}^{n-d} \oplus W$$

where $[0, \infty)^{(j)} = \{(z = (z_1, \ldots, z_d) \in [0, \infty)^d \mid z_j = 0\}$. The tangent space to $\mathbb{R}^m \oplus \mathcal{F}^j$ at the solution (λ, x) is equal to

$$T_{(\lambda,x)}(\mathbb{R}^m \oplus \mathcal{F}^j) = \mathbb{R}^m \oplus \mathbb{R}^{(j)} \oplus \mathbb{R}^{n-d} \oplus W$$

where the subspace $\mathbb{R}^{(j)}$ consists of those points $z \in \mathbb{R}^d$ whose jth coordinate is equal to 0. Moreover, $T^{\partial}_{(\lambda,x)}(\mathbb{R}^m \oplus \mathcal{F}^j)$ is equal to the subspace

$$T^{\partial}_{(\lambda,x)}(\mathbb{R}^m \oplus \mathcal{F}^j) = \mathbb{R}^m \oplus \mathbb{R}^{\Sigma} \oplus \mathbb{R}^{n-d} \oplus W.$$

We shall prove that the linearization of the map $F|\mathbb{R}^m \oplus \mathcal{F}^j : \mathbb{R}^m \oplus \mathcal{F}^j \to \mathbb{R}^N \oplus W$ is surjective at the solution (λ, x) of $F(\lambda, x) = 0$. In order to do so, we choose $y \in \mathbb{R}^N \oplus W$. We already know that the linearization $F'(\lambda, x) : \mathbb{R}^m \oplus \mathbb{R}^d \oplus \mathbb{R}^{n-d} \oplus W \to \mathbb{R}^N \oplus W$ is surjective and so there exists $(\delta\lambda, \delta u) \in \mathbb{R}^m \oplus (\mathbb{R}^d \oplus \mathbb{R}^{n-d} \oplus W) \to \mathbb{R}^N \oplus W$ solving $F'(\lambda, x)(\delta\lambda, \delta u) = y$. We also know that the kernel N' of $F'(\lambda, x)$ is transversal to $\mathbb{R}^m \oplus \{0\}^d \oplus \mathbb{R}^{n-d} \oplus W$ in $\mathbb{R}^m \oplus \mathbb{R}^d \oplus \mathbb{R}^{n-d} \oplus W$. Therefore, we have the representation $(\delta\lambda, \delta u) = (\delta\lambda_1, \delta u_1) + (\delta\lambda_2, \delta u_2)$ where $(\delta\lambda_1, \delta u_1) \in N'$ and $(\delta\lambda_2, \delta u_2) \in \mathbb{R}^m \oplus \{0\}^d \oplus \mathbb{R}^{n-d} \oplus W$. Consequently, $(\delta\lambda_2, \delta u_2)$ belongs to $T_{(\lambda,x)}(\mathbb{R}^m \oplus \mathcal{F}^j)$ and satisfies $F'(\lambda, x)(\delta\lambda_2, \delta x_2) = y$ as we wanted to prove.

Next we shall show that the kernel of the linearization $F'(\lambda, x)|T_{(\lambda,x)}(\mathbb{R}^m \oplus \mathcal{F}^j)$ is transversal to the subspace $T^{\partial}_{(\lambda,x)}(\mathbb{R}^m \oplus \mathcal{F}^j)$ in $T_{(\lambda,x)}(\mathbb{R}^m \oplus \mathcal{F}^j)$. To do so we choose $(\delta\lambda, \delta u) \in T_{(\lambda,x)}(\mathbb{R}^m \oplus \mathcal{F}^j) = \mathbb{R}^m \oplus \mathbb{R}^{(j)} \oplus \mathbb{R}^{n-d} \oplus W$ and have the representation $(\delta\lambda, \delta u) = (\delta\lambda_1, \delta u_1) + (\delta\lambda_2, \delta u_2)$ where $(\delta\lambda_1, \delta u_1) \in N'$ and $(\delta\lambda_2, \delta u_2) \in \mathbb{R}^m \oplus \{0\}^d \oplus \mathbb{R}^{n-d} \oplus W \subset T^{\partial}_{(\lambda,x)}(\mathbb{R}^m \oplus \mathcal{F}^j)$. Moreover, denoting by

$$p : \mathbb{R}^m \oplus \mathbb{R}^d \oplus \mathbb{R}^{n-d} \oplus W \to \mathbb{R}^d$$

the projection, we conclude that $p(\delta\lambda_1, \delta u_1) = p(\delta\lambda, \delta u) = a \in \mathbb{R}^d$ where the jth coordinate of a is equal to 0. Hence $(\delta\lambda_1, \delta u_1) \in T_{(\lambda,x)}(\mathbb{R}^m \oplus \mathcal{F}^j)$ as we wanted to show. The same way one verifies the more general claim in (b).

(c) To finish the proof of Lemma 5.23 we carry out the above construction near every point $(0, x)$ for $x \in \mathcal{M}^{f+t}$. Since the solution set $\mathcal{M}^{f+t}$ is compact, we can select finitely many neighborhoods $U(0, x_j)$, $1 \leq j \leq m$, covering $\{0\} \oplus \mathcal{M}^{f+t}$. Now taking as a perturbation the sum of all the finitely many sections constructed in these neighborhoods, we obtain the section

$$F(\lambda, x) = (f + t)(x) + \sum_{j=1}^{l} \lambda_j s_j(x)$$

of the bundle $Y^1 \to \mathbb{R}^l \oplus X^1$ which is a Fredholm section in view of Theorem 3.9. We claim that there exists a sufficiently small $\varepsilon > 0$ such that

$$\overline{S}_\varepsilon = \{(\lambda, x) \in \mathbb{R}^l \oplus X^1 | F(\lambda, x) = 0 \text{ and } |\lambda| \leq \varepsilon\} \subset \bigcup_{j=1}^{n} U(0, x_j).$$

Indeed, otherwise there exists a sequence $(\lambda^k, x^k) \in \mathbb{R}^l \oplus X^1$ satisfying $F(\lambda^k, x^k) = 0$ and $|\lambda^k| \leq \frac{1}{k}$ but $(\lambda^k, x^k) \notin \bigcup_{j=1}^n U(0, x)$. Since the supports of t and s_j are contained in U, we conclude that $x^k \in U$ for all k. From $f(x^k) = -t(x^k) - \sum_{j=1}^l \lambda_j^n s_j(x^k)$ we find $N(f(x^k)) \leq 1$ for k large and since the set U has the property (ii) of Theorem 5.12, we may assume that the sequence (x^k) converges to the point $x \in \overline{U}$. Because $(f + t)(x) = 0$, we have $(0, x) \in \{0\} \oplus M^{f+t} \subset \bigcup_{j=1}^m U(0, x_j)$. This contradicts our assumption $(\lambda^k, x^k) \notin \bigcup_{j=1}^m U(0, x_j)$ for all k. Consequently, the section F has the properties (i) and (ii) of Lemma 5.23 if $\varepsilon > 0$ is sufficiently small.

It remains to show that the solution set $\overline{S}_\varepsilon$ is compact in $\mathbb{R}^l \oplus U$ if $\varepsilon > 0$ is sufficiently small. To see this we take the solution $(\lambda, x) \in \overline{S}_\varepsilon$, then $f(x) = -t(x) - \sum_{j=1}^l \lambda_j s_j(x)$. Since $t \in \frac{1}{2}\mathcal{O}$ and $N(s_j(x)) \leq 1$ for $1 \leq j \leq l$ and all $x \in X$, we conclude from $|\lambda| \leq \varepsilon$ that $N(f(x)) \leq 1$ if ε is sufficiently small. In addition, since $\mathcal{M}^f \subset U$ and since the supports of t and s_j are contained in U, it follows that $x \in U$. Hence if $(\lambda^k, x^k) \in \overline{S}_\varepsilon$, then $x^k \in U$ so that by Theorem 5.12, the sequence (x^k) possesses a convergent subsequence. Since $|\lambda^k| \leq \varepsilon$, we conclude that the sequence (λ^k, x^k) has a subsequence converging to a solution (λ, x) of $F(\lambda, x) = 0$. The solution belongs to $\overline{B}_\varepsilon \oplus U$. Indeed, if $x \in X \setminus U$, then $t(x) + \sum_{j=1}^l \lambda_j s_j(x) = 0$ since the supports of t

and s_j are contained in U. Hence $f(x) = 0$ which implies $x \in f^{-1}(0) \subset U$, contradicting the assumption $x \in X \setminus U$. Consequently, the solution set $\overline{S}_\varepsilon$ is compact in $\mathbb{R}^l \oplus X$ and $\overline{S}_\varepsilon \subset \overline{B}_\varepsilon \oplus U$. The compactness in $\mathbb{R}^l \oplus X_\infty$ follows arguing as in Theorem 5.11. The proof of Lemma 5.23 is complete. ■

Abbreviate by S_ε the set consisting of those points (λ, x) in $\overline{S}_\varepsilon$ for which $|\lambda| < \varepsilon$.

Lemma 5.24. *The Fredholm section F of the fillable strong bundle $Y^1 \to B_\varepsilon \oplus X^1$ introduced in Lemma 5.23 is in good position to the corner structure of $B_\varepsilon \oplus X$. Consequently, $F^{-1}(0)$ is a smooth manifold with boundary with corners.*

Proof. By Lemma 5.23, the linearization $F'(\lambda, x)$ is surjective and its kernel is transversal to $T^\partial_{\lambda,x}(\mathbb{R}^l \oplus X)$ in $T_{(\lambda,x)}(\mathbb{R}^l \oplus X)$ at every point $(\lambda, x) \in S_\varepsilon$. Arguing as in the proof of Lemma 5.23, the kernel of $F'(\lambda, x)$ is neat in $T_{(\lambda,x)}(\mathbb{R}^l \oplus X)$ at every point $(\lambda, x) \in S_\varepsilon$. This implies that the kernel of $F'(\lambda, x)$ is in good position with respect to the corner structure of $\mathbb{R}^l \oplus X$ at (λ, x). Consequently, arguing as in the proof Theorem 5.16, the solution set $S_\varepsilon = F^{-1}(0)$ carries a structure of a smooth manifold with boundary with corners. ■

Continuing with the proof of Theorem 5.22 we know from the proof of Lemma 5.23 that the solution set S_ε satisfies

$$S_\varepsilon = \{(\lambda, x) \in \mathbb{R}^l \oplus X^1 | F(x, \lambda) = 0 \text{ and } |\lambda| < \varepsilon\} \subset \bigcup_{j=1}^n U(0, x_j)$$

where F is the Fredholm section introduced in Lemma 5.23.

Every point $(0, x_j)$, belongs to $d(x_j)$-many faces $\mathbb{R}^l \oplus \mathcal{F}^k_{x_j}$ for $1 \le k \le d(x_j)$. If $1 \le j \le n$ and if $\sigma \subset \{1, \ldots, d(x_j)\}$, we define

$$S^{j,\sigma}_\varepsilon := S_\varepsilon \cap U(0, x_j) \cap \big(\mathbb{R}^l \oplus \bigcap_{k \in \sigma} \mathcal{F}^k_{x_j}\big).$$

In view of Lemma 5.23 and arguing as in Theorem 5.16, the set, $S^{j,\sigma}_\varepsilon$ is a manifold having a boundary with corners. We introduce the projections $\beta : S_\varepsilon \to \mathbb{R}^l$ and $\beta_{j,\sigma} : S^{j,\sigma}_\varepsilon \to \mathbb{R}^l$ as the compositions

$$\beta : S_\varepsilon \xrightarrow{j} \mathbb{R}^l \oplus X \xrightarrow{\pi} \mathbb{R}^l$$

$$\beta_{j,\sigma} : S^{j,\sigma}_\varepsilon \xrightarrow{j} \mathbb{R}^l \oplus X \xrightarrow{\pi} \mathbb{R}^l$$

where j is the injection mapping and π is the projection $\pi(\lambda, x) = \lambda$.

In view of Sard's theorem, the set of regular values of projections β and $\beta_{j,\sigma}$ has full measure in B_ε. Denote by λ^* a common regular value of all these projections and introduce the sc$^+$-section

$$s^* = t + \sum_{j=1}^{l} \lambda_j^* s_j$$

so that

$$(f + s^*)(x) = F(\lambda^*, x).$$

We claim that at every solution x of $(f + s^*)(x) = 0$, the linearization $(f + s^*)'(x)$ is surjective and that the kernel of $(f + s^*)'(x)$ is transversal to $T_x^\partial X$ in $T_x X$.

In order to prove the claim we take a solution x of $(f + s^*)(x) = 0$ and first prove that $(f + s^*)'(x)$ is surjective. We have $F(\lambda^*, x) = 0$ so that $(\lambda^*, x) \in S_\varepsilon$. By Lemma 5.23, the linearization $F'(\lambda^*, x)$ is surjective. Hence given $y \in Y^1$, there exists $(\delta\lambda, \delta x) \in T_{(\lambda,x)}(\mathbb{R}^l \oplus X)$ solving $F'(\lambda^*, x)(\delta\lambda, \delta x) = y$. The point λ^* is a regular value of the projection $\beta : S_\varepsilon \to \mathbb{R}^l$ and the tangent space to S_ε at (λ^*, x) coincides with the kernel N of $F'(\lambda^*, x)$. Hence there exists $(\delta\lambda_1, \delta x_1) \in N$ solving the equation $d\beta(\lambda^*, x)(\delta\lambda_1, \delta x_1) = \delta\lambda$ and satisfying

$$F'(\lambda^*, x)(\delta\lambda_1, \delta x_1) = 0.$$

From $d\beta(\lambda^*, x)(\delta\lambda_1, \delta x_1) = \delta\lambda_1$ we conclude $\delta\lambda_1 = \delta\lambda$ so that

$$F'(\lambda^*, x)(\delta\lambda, \delta x_1) = 0.$$

Consequently,

$$\begin{aligned} y &= F'(\lambda^*, x)(0, \delta x - \delta x_1) \\ &= D_2 F(\lambda^*, x)(\delta x - \delta x_1) \\ &= (f + s^*)'(\delta x - \delta x_1) \end{aligned}$$

showing that the linearization $(f + s^*)'(x)$ is indeed surjective.

Next we show that the kernel of the linearization $(f + s^*)'(x)$ is transversal to $T_x^\partial X$ at every solution x of $(f + s^*)(x) = 0$. Since $F(\lambda^*, x) = 0$ and $F^{-1}(0) \subset \bigcup_{1 \leq j \leq n} U(0, x_j)$, the point (λ^*, x) belongs to some open neighborhood $U(0, x_j)$. To prove the transversality we work in local coordinates as in the proof of Lemma 5.23. We assume that F is already filled and is the form

$$F : \mathbb{R}^l \oplus [0, \infty)^d \oplus \mathbb{R}^{n-d} \oplus W \to \mathbb{R}^N \oplus W.$$

The integer d is equal to the order $d = d(x_j)$ of x_j. In these coordinates the point x_j corresponds to the point $0 \in [0, \infty)^d \oplus \mathbb{R}^{n-d} \oplus W$ so that (λ^*, x_j) corresponds to $(\lambda^*, 0) \in \mathbb{R}^l \oplus [0, \infty)^d \oplus \mathbb{R}^{n-d} \oplus W$. The point corresponding to (λ^*, x) we shall denote again by (λ^*, x). Its degree $d(x)$ satisfies $d(x) \leq d$. We use the notation $X = [0, \infty)^d \oplus \mathbb{R}^{n-d} \oplus W$. If $\Sigma = \{i \in \{1, \ldots, d\} \mid x_i = 0\}$, the tangent space $T_{(\lambda^*, x)}(\mathbb{R}^l \oplus \bigcap_{j \in \Sigma} \mathcal{F}^j))$ is equal to

$$T_{(\lambda^*, x)}(\mathbb{R}^l \oplus \bigcap_{j \in \Sigma} \mathcal{F}^j) = \mathbb{R}^l \oplus \mathbb{R}^\Sigma \oplus \mathbb{R}^{n-d} \oplus W.$$

Here $\mathcal{F}^j$ for $j \in \Sigma$ are the faces containing the point x. By Lemma 5.23, the kernel N of the linearization $F'(\lambda^*, x)$ is transversal to $T^{\partial}_{(\lambda^*, x)}(\mathbb{R}^l \oplus X)$. Hence, given $\delta x \in T_x X$, there exists $(\delta\lambda_1, \delta x_1)$ belonging to the kernel N of $F'(\lambda^*, x)$ and $(\delta\lambda_2, \delta x_2)$ in $T^{\partial}_{(\lambda^*, x)}(\mathbb{R}^l \oplus X)$ such that

$$(\delta\lambda_1, \delta x_1) + (\delta\lambda_2, \delta x_2) = (0, \delta x).$$

The point λ^* is a regular value of the projection $\beta_{j,\Sigma} : S^{j,\Sigma}_{\varepsilon} \to \mathbb{R}^l$. Since $d\beta_{j,\Sigma}(\lambda^*, x)$ is defined on $N \cap T_{(\lambda^*, x)}(\mathbb{R}^l \oplus \bigcap_{j \in \Sigma} \mathcal{F}^j) = N \cap (\mathbb{R}^l \oplus \mathbb{R}^\Sigma \oplus \mathbb{R}^{n-d} \oplus W)$ we find $(\delta\lambda', \delta x')$ belonging to $N \cap (\mathbb{R}^l \oplus \mathbb{R}^\Sigma \oplus \mathbb{R}^{n-d} \oplus W)$ such that $d\beta_{j,\Sigma}(\lambda^*, x)(\delta\lambda', \delta x') = \delta\lambda_1$. Using $d\beta_{j,\Sigma}(\lambda^*, x)(\delta\lambda', \delta x') = \delta\lambda_1$, we obtain

$$F'(\lambda^*, x)(0, \delta x_1 - \delta x') = D_2 F(\lambda^*, x)(\delta x_1 - \delta x') = 0$$

and

$$(\delta x_1 - \delta x') + (\delta x_2 + \delta x') = \delta x.$$

Since $\delta x_2 + \delta x'$ belongs to $\mathbb{R}^l \oplus \mathbb{R}^\Sigma \oplus \mathbb{R}^{n-d} \oplus W$, the kernel of the linearization $(f + s^*)'(x)$ is indeed transversal to $T^{\partial}_x X$ in $T_x X$ as we wanted to show.

In order to finish the proof of Theorem 5.22 we observe that so far we have proved that the Fredholm section $f + s^*$ is in general position to the boundary ∂X according to Definition 5.17. Consequently, in view of Theorem 5.18, the solution set $\mathcal{M}^{f+s^*}$ is a compact manifold with boundary with corners. This completes the proof of Theorem 5.22. ■

The following result is proved along the lines of the previous result. In contrast to Theorem 5.22 we impose conditions on the Fredholm sections at those solutions which are located at the boundary ∂X, while the perturbation has its support away from the boundary.

Theorem 5.25. *Let $p : Y \to X$ be a fillable strong M-polyfold bundle having a nonempty boundary ∂X and let f be a proper Fredholm section of p. We assume that the sc-structure on X is built on separable sc-Hilbert spaces and assume that U is an open neighborhood of the solution set $\mathcal{M}^f = f^{-1}(0)$. Moreover, let N be an auxiliary norm guaranteed by Theorem 5.12. If at every point $x \in \partial X$ solving $f(x) = 0$ the linearization of f is surjective and the kernel of the linearization is in good position to the corner structure of ∂X, then there exists an arbitrarily small sc^+-section s which has its support in U and which vanishes near ∂X so that the Fredholm section $f + s$ is in good position as defined in Definition 5.15. In particular, the solution set $\mathcal{M}^{f+s} = (f + s)^{-1}(0)$ is a smooth compact manifold with boundary with corners.*

The proof goes as follows. By the assumption on the solutions of $f(x) = 0$ at the boundary ∂X, the solution set of $f = 0$ admits a good parametrization near the boundary points. Then one finds enough sc^+-sections which vanish near the boundary and have their supports in U so that the Fredholm section $F(\lambda, x) = f(x)+\sum_{i=1}^k \lambda_i s_i(x)$ has, at every solution $(0, x)$ of $F(0, x) = 0$ with $x \in U \setminus \partial X$, a linearization which is surjective. From the assumption of the theorem it follows that the linearization $F'(0, x)$ is automatically surjective also at the boundary points $x \in \partial X$ satisfying $f(x) = 0$ and hence $F(0, x) = 0$, in addition, its kernel is again, by assumption, in good position to the corner structure of $\mathbb{R}^k \oplus X$. Therefore, as in the proof of Theorem 5.22, one finds a suitable ε so that the solutions space $S_\varepsilon = \{(\lambda, x) | \, F(\lambda, x) = 0, |\lambda| < \varepsilon\}$ is a smooth manifold with boundary with corners. A small regular value λ^* of the projection

$$(\lambda, x) \mapsto \lambda$$

defines the section $s^* = \sum_{i=1}^k \lambda_i^* s_i$ for which $f + s^*$ has the desired properties. This completes the proof of Theorem 5.25.

5.5 Some Invariants for Fredholm Sections

The following discussion extends some standard material from the classical nonlinear Fredholm theory to the M-polyfold context. We introduce the notion of an sc-differential k-form starting with comments about our notation. For us the tangent bundle of X is $TX \to X^1$, that is, it is only defined for the base points in X^1. An **sc-vector field** on X is an sc-smooth section A of the tangent bundle $TX \to X^1$ and hence it is defined on X^1. Similarly, an

sc-differential form on X which we will define next, is only defined over the base points in X^1. The definition of a vector field and the following definition of an sc-differential form are justified since the construction of TX, though only defined over X^1, requires the knowledge of X. If X is a M-polyfold, then $\oplus_k TX_\infty$ denotes the k-fold Whitney sum of k copies of the tangent space TX .

Definition 5.26. *An* **sc-differential *k*-form on the** M**-polyfold** X *is an sc-smooth map* $\omega : \oplus_k TX \to \mathbb{R}$ *which is linear in each argument separately, and skew symmetric.*

If ω is an sc-differential form on X, we may also view it as an sc-differential form on X^i. Denote by $\Omega^*(X^i)$ the graded commutative algebra of sc-differential forms on X^i. Then we have the inclusion map

$$\Omega^*(X^i) \to \Omega^*(X^{i+1}).$$

which is injective since X_{i+1} is dense in X_i and the forms are sc-smooth. Hence we have a directed system whose direct limit is denoted by $\Omega^*_\infty(X)$. An element ω of degree k in $\Omega^*_\infty(X)$ is a skew-symmetric map $\oplus_k (TX)_\infty \to \mathbb{R}$ such that it has an sc-smooth extension to an sc-smooth k-form $\oplus_k TX^i \to \mathbb{R}$ for some i. We shall refer to an element of $\Omega^k_\infty(X)$ as an sc-smooth differential form on X_∞. We note, however, that it is part of the structure that the k-form is defined and sc-smooth on some X^i.

Next we associate with an sc-differential k-form ω its exterior differential $d\omega$ which is a $(k+1)$-form on the M-polyfold X. Let $A_0, \ldots, A_k$ be $k+1$ many sc-smooth vector fields on X. We define $d\omega$ on X^1, using the familiar formula, by

$$\begin{aligned} d\omega(A_0, \ldots, A_k) &= \sum_{i=0}^{k} (-1)^i D(\omega(A_0, \ldots, \widehat{A}_i, \ldots, A_k)) \cdot A_i \\ &\quad + \sum_{i<j} (-1)^{(i+j)} \omega([A_i, A_j], A_0, \ldots, \widehat{A}_i, \ldots, \widehat{A}_j, \ldots, A_k). \end{aligned}$$

The right-hand side of the formula above only makes sense at the base points $x \in X_2$. This explains why $d\omega$ is a $(k+1)$-form on X^1. By the previous discussion the differential d defines a map

$$d : \Omega^k(X^i) \to \Omega^{k+1}(X^{i+1})$$

and consequently induces a map

$$d : \Omega^*_\infty(X) \to \Omega^{*+1}_\infty(X)$$

having the usual property $d^2 = 0$. Then $(\Omega^*_\infty(X), d)$ is a graded differential algebra which we shall call the de Rham complex.

If $\varphi : M \to X$ is an sc-smooth map from a finite-dimensional manifold M into an M-polyfold X, then it induces an algebra homomorphism

$$\varphi^* : \Omega^*_\infty(X) \to \Omega^*(M)_\infty$$

satisfying

$$d(\varphi^*\omega) = \varphi^* d\omega.$$

Since $d^2 = 0$, we can define as usual the deRham cohomology groups

$$H^*_{dR}(X, \mathbb{R}) = \oplus_{i=0}^\infty H^i(X, \mathbb{R}).$$

A differential form ω is in the following a finite formal sum of forms of possibly different degrees $\omega = \omega_0 + \ldots + \omega_n$.

Next we consider a fillable strong M-polyfold bundle $p : Y \to X$ in which the local models of X are built on separable sc-Hilbert spaces. We assume that $\partial X = \emptyset$. Let $f : X \to Y$ be a proper Fredholm section of the bundle p. Let U be the open neighborhood of the solution set $f^{-1}(0)$ and $\mathcal{O} \subset \Gamma^+(p)$ the space of sections as in Theorem 5.21. In view of this theorem, there exist small sections $s \in \mathcal{O}$ having their supports in U and having the property that $f + s$ are proper Fredholm sections of the bundle $p^1 : Y^1 \to X^1$. In addition, the linearizations of $f+s$ at the solution set $\mathcal{M}^{f+s} = \{x \in X \,|\, (f+s)(x) = 0\}$ are surjective.

Definition 5.27. *A perturbation $s \in \mathcal{O}$ is called* **generic***, if the Fredholm section $f + s$ is in general position, i.e. the linearization is surjective at every zero of $f + s$ and, in addition, the kernel of the lineraization is in good position to the corner structure at the zeros belonging to the boundary.*

Finally, we assume that the given section f is orientable and let $\mathfrak{o}$ be an orientation of f as defined in Appendix 6.4. Then a map

$$\Phi_{(f,\mathfrak{o})} : H^*_{dR}(X, \mathbb{R}) \to \mathbb{R}$$

can be defined as follows. We take a generic $s \in \mathcal{O}$. Then the associated solution set $\mathcal{M}^{f+s}$ is contained in X_∞ and, in view of Theorem 5.14, is a

compact smooth manifold without boundary which, in addition, is oriented since $(f, \mathfrak{o})$ is an oriented Fredholm section. Hence we can integrate a differential form $\omega = \omega_0 + \omega_1 + \cdots + \omega_n$ over $\mathcal{M}^{f+s}$ where we put the integral equal to zero on each component of $\mathcal{M}^{f+s}$ if the degrees don't match the local dimensions. This way one obtains the real number

$$\int_{\mathcal{M}^{f+s}} \omega := \sum_{i=1}^{n} \int_{\mathcal{M}^{f+s}} j^* \omega_i$$

with the sc-smooth inclusion mapping $j : \mathcal{M}^{f+s} \to X_\infty$.

If $s' \in \mathcal{O}$ is a second generic perturbations of f, we find a generic sc-smooth homotopy $s_t \in \Gamma^+(p)$ connecting $s_0 = s$ with $s_1 = s'$ so that $F(t, x) = f(x) + s_t(x)$ is a proper Fredholm section of the bundle $Y^1 \to [0, 1] \times X^1$ transversal to the zero section $\mathcal{M}^F = \{(t, x) \in [0, 1] \times X^1 | \, F(t, x) = 0\}$.

This can be seen as follows. The two sc$^+$-sections s_i for $i = 0$ and $i = 1$ have their supports comtained in U and $f + s_i$ are proper Fredholm sections of the bundle $p^1 : Y^1 \to X^1$. In addition, $N(s_i) < 1$ and the linearizations $(f + s_i)'(x)$ are surjective at every x belonging to the solution set $\mathcal{M}^{f+s_i}$. For every $t \in [0, 1]$, the section $(1 - t)s_0 + ts_1$ is of class sc$^+$ having its support in U and satisfying $N((1 - t)s_0 + ts_1) < 1$. Consider the Fredholm section $\bar{f}(t, x) = f(x) + (1 - t)s_0(x) + ts_1(x)$ of the bundle $Y^1 \to [0, 1] \times X^1$. Clearly, the section $\bar{f}$ is proper since f is proper and $[0, 1]$ is compact. It vanishes at the points $(t, x) \in \{i\} \times \mathcal{M}^{f+s_i}$ for $i = 0$ and $i = 1$. At these points linearization of $\bar{f}$ is surjective because the linearizations of $(f + s_i)$ are surjective at points x belonging to $\mathcal{M}^{f+s_i}$ for $i = 0$ and $i = 1$. Moreover, at every such point the kernel of the linearization is good position to the corner structure of $\partial([0, 1] \times X)$. Then one finds a finite number of sc$^+$-sections $\bar{s}_1, \ldots \bar{s}_k$ which vanish near the boundary $\partial([0, 1] \times X)$ and have their supports in $[0, 1] \times U$so that the Fredholm section

$$F(\lambda, t, x) = \bar{f}(t, x) + \sum_{i=1}^{k} \lambda_i \bar{s}_i(t, x)$$

has at every zero $(0, t, x)$ of $F(0, t, x) = 0$ with $t \in (0, 1)$ a surjective linearization. At every zero of the form $(0, 0, x)$ or $(0, 1, x)$ of $F(0, t, x) = 0$, the linearization is also surjective since the linearization $(f + s_i)'(x)$ is surjective for $i = 0$ and $i = 1$. In addition, at every solution $(0, 0, x)$ or $(0, 1, x)$, the kernel of the linearization of F is in good position the corner structure of

$\mathbb{R}^k \oplus [0,1] \oplus X$. Therefore, we find a suitable ε so that the solution space $S_\varepsilon = \{(\lambda, t, x) | \, F(\lambda, t, x) = 0, |\lambda| < \varepsilon\}$ is a smooth manifold with boundary with corners. A small regular value λ^* of the projection

$$(\lambda, t, x) \mapsto \lambda$$

defines the section $s(t,x) = (1-t)s_0(x) + ts_1(x) + \sum_{i=1}^k \lambda_i^* \bar{s}_i(t,x)$ having the desired properties.

The manifold $\mathcal{M}^F$ is a compact smooth manifold whose boundary is given by $\partial\mathcal{M}^F = \mathcal{M}^{f+s'} \cup (-\mathcal{M}^{f+s})$. With the projection $\pi : [0,1] \times X \to X$ we obtain for the closed form ω by Stokes' theorem

$$0 = \int_{\mathcal{M}^F} d(\pi^*\omega) = \int_{\mathcal{M}^{f+s'}} \omega - \int_{\mathcal{M}^{f+s}} \omega.$$

One concludes that the number $\int_{\mathcal{M}^{f+s}} \omega$ does not depend on the choice of a generic s. Therefore, we can define $\Phi_{(f,\mathfrak{o})}$ by

$$\Phi_{(f,\mathfrak{o})}([\omega]) = \int_{\mathcal{M}^{f+s}} \omega,$$

where s is any generic small perturbation of the proper Fredholm section f.

Consider two oriented proper Fredholm sections $(f_0, \mathfrak{o}_0)$ and $(f_1, \mathfrak{o}_1)$ of the above M-polyfold bundle $p : Y \to X$, and a proper homotopy f_t of Fredholm sections connecting f_0 with f_1 such that the map $(x,t) \mapsto f(t,x) = f_t(x)$ is a proper Fredholm section of the bundle $Y \to [0,1] \times X$. The homotopy is called **oriented** if there exists an orientation $\overline{\mathfrak{o}}$ inducing the given orientations $(-\mathfrak{o}_0) \cup \mathfrak{o}_1$ at the ends. Then an argument as above shows that

$$\Phi_{(f_0,\mathfrak{o}_0)} = \Phi_{(f_1,\mathfrak{o}_1)}.$$

In order to relate this to the usual mapping degree of f we assume that $(f, \mathfrak{o})$ is an oriented proper Fredholm section whose Fredholm index is equal to 0. If $s \in \mathcal{O}$ is a generic perturbation, then $f + s$ is also a proper Fredholm section whose Fredholm index is equal to 0 (by Theorem 3.9) and which, in addition, is transversal to the zero section. Hence, if $x \in \mathcal{M}^{f+s}$ then the linearization $(f+s)'(x)$ is surjective and injective and the local analysis of the zero set of a Fredholm map (Theorem 4.18 and Theorem 4.6) shows that the solution x is isolated. In view of the compactness, the zero set $\mathcal{M}^{f+s} = \{x_1, \ldots, x_k\}$ consists of finitely many points. Hence we can define the degree of f by

$$\deg(f, \mathfrak{o}) = \Phi_{(f,\mathfrak{o})}([1]).$$

where [1] is the cohomology of the constant function equal to 1 and where the integration of [1] over $\mathcal{M}^{f+s}$ is the signed sum over the finitely many points in $\mathcal{M}^{f+s}$. A point $x_i \in \mathcal{M}^{f+s}$ counts as $+1$ if its orientation $\mathfrak{o}$ agrees with the natural orientation of the isomorphism $(f+s)'(x_i)$ as defined in Appendix 6.4, and -1 otherwise.

The degree is an invariant of oriented proper Fredholm sections under oriented proper homotopies as the above discussion shows.

6 Appendix

We shall first study subspaces which are in good position to a partial quadrant according to Definition 4.10.

6.1 Two Results on Subspaces in Good Position

We consider the sc-Banach space $E = \mathbb{R}^n \oplus W$ containing the partial quadrant $C = [0, \infty)^n \oplus W$. Our aim is to prove the following proposition.

Proposition 6.1. *If the finite dimensional linear subspace N of the sc-Banach space E is in good position to the partial quadrant C in E, then $C \cap N$ is a partial quadrant in N.*

In higher dimensions one easily can construct a subspace N for which $C \cap N$ has a nonempty interior, but $C \cap N$ is not a partial quadrant.

Next we prove Proposition 4.11 restated as Proposition 6.2.

Proposition 6.2. *If $N \subset E$ is neat with respect to the partial quadrant C, then N is in good position to C and $C \cap N$ is a partial quadrant in N.*

Proof. By assumption, the subspace N possesses an sc-complement $N^\perp$ in E which is contained in C. This implies that $N^\perp = \{0\} \oplus Q$ for some sc-subspace Q in W. Since $E = N \oplus N^\perp$, we find a point of the form $(1, 1, 1, .., 1, e) \in N$, which implies that $N \cap C$ has a nonempty interior. Next we take $c = 1$ and assume that $(n, m) \in N \oplus N^\perp$ satisfies the estimate $\|m\| \leq \|n\|$. Then $n + m \in C$ if and only if $n = 2[\frac{n+m}{2} + \frac{-m}{2}] \in C$, because C is convex, $\mathbb{R}^+ \cdot C = C$, and $-m \in C$. Hence N is in good position to the partial quadrant C and, by Proposition 6.1, the subset $C \cap N$ is a partial quadrant in N. The proof of Proposition 6.2 is complete. ■

6.2 Quadrants and Cones

A **closed convex cone** P in a finite-dimensional vector space N is a closed convex subset so that $P \cap (-P) = \{0\}$ and $\mathbb{R}^+ \cdot P = P$. An **extreme ray** in a closed convex cone P is a subset R of the form

$$R = \mathbb{R}^+ x$$

where $x \in P \setminus \{0\}$ so that if $y \in P$ and $x - y \in P$, then $y \in R$. If the cone P has a nonempty interior it generates the vector space N, i.e., $N = P - P$.

The following version of the Krein-Milman theorem is well-known. See Exercise 30 on page 72 in citeSchaefer.

Lemma 6.3. *A closed convex cone P in a finite-dimensional vector space N is the closed convex hull of its extreme rays.*

Proof. Take a hyperplane A in N such that $A \cap P = \{0\}$. Choose a point $a \in P \setminus \{0\}$ and define the affine subspace $A' = a + A$. Note that for every $x \in P \setminus \{0\}$, there is $t > 0$ such that $tx \in P \cap A'$. Indeed, if $x \in P \setminus \{0\}$, then $x = sa + y$ for some $s \in \mathbb{R}$ and $y \in A$. If $s = 0$, then $x = y \in P \cap A = \{0\}$ contradicting $x \neq 0$. . Hence $s \neq 0$. If $s < 0$, then $x + (-s)a = y$ and since x and $(-s)a \in P$, we conclude that $y \in P \cap A = \{0\}$. So, $x = sa$ with $s < 0$ contradicting that $x \in P \setminus \{0\}$. The set $P \cap A'$ is convex and closed. It is also bounded since if $x_n \in P \cap A'$ and $\|x_n\| \to \infty$, then $x_n = a + y_n$ with $x_n \in A$ satisfying $\|y_n\| \to \infty$. We may assume that $y_n/\|y_n\| \to y \in A$. Then $x_n/\|y_n\| \in P$ and $x_n/\|y_n\| \to y$. Hence $y \in P \cap A = \{0\}$ contradicting $\|y\| = 1$. By the Krein-Milman theorem applied to the set $K = P \cap A'$, the set K is equal to the closure of the convex hull of the extreme points of K. Hence to prove the lemma it suffices to show that an extreme point x of K generates an extreme ray $R = \mathbb{R}^+ \cdot x$ of P. To see this take $y \in P \setminus \{0\}$ such that $x - y \in P$. We have to show that $y = t \cdot x$ for some $t > 0$. If $x = y$, then we are done. Otherwise, $y = ta + y'$ with $t > 0$ and $y \in A$, and $x - y = sa + y''$ with $s > 0$ and $y'' \in A$. Then $x = y + (x - y) = (t + s)a + (y' + y'') \in K$ and therefore $s + t = 1$. By assumption the point x is an extreme point of K. Consequently, since x can be written as

$$x = t\left(\frac{1}{t}y\right) + s\left(\frac{1}{s}(x - y)\right)$$

with the points $\frac{1}{t}y = a + \frac{1}{t}y'$ and $\frac{1}{s}(x - y) = a + \frac{1}{s}y''$ belonging to K, it follows that $x = \frac{1}{t}y = \frac{1}{s}(x - y)$. Hence $y = t \cdot x$ as claimed. ∎

We observe that a quadrant in N has precisely $\dim(N)$ many extreme rays. A closed convex cone P is called **finitely generated** provided P has finitely many extreme rays. In that case P is the convex hull of its finitely many extreme rays. For example, if C is a partial quadrant in E and $N \subset E$ is a finite-dimensional subspace of E so that $C \cap N$ is a closed convex cone, then $C \cap N$ is finitely generated.

Lemma 6.4. *Let N be a finite-dimensional vector space and $P \subset N$ a closed convex cone with nonempty interior. Then P is a quadrant if and only if it has* $\dim(N)$*-many extreme rays.*

Proof. Assume that P has $\dim(N)$-many extreme rays, $R_1 = \mathbb{R}^+ \cdot x_j, \ldots, R_d = \mathbb{R}^+ \cdot x_d$ where $d = \dim N$. In view of Lemma 6.3, the cone P is the closed convex hull of $R_1, \ldots, R_d$. Since P has nonempty interior, the vectors $x_1, \ldots, x_d$ are linearly independent. Take the linear isomorphism $F: N \to \mathbb{R}^d$ mapping x_j to the standard vector e_j. Then $F(P)$ is the standard quadrant in $\mathbb{R}^d$. The converse is proved similarly. ■

If $a \in C = [0, \infty)^n \oplus W \subset \mathbb{R}^n \oplus W$, we have the representation $a = (a_1, \ldots, a_n, a_\infty)$ where $(a_1, \ldots, a_n) \in \mathbb{R}^n$ and $a_\infty \in W$. By σ_a we shall denote the collection of all indices $i \in \{1, \ldots, n\}$ for which $a_i = 0$ and denote the complementary set of indices in $\{1, \ldots, n\}$ by σ_a^c. Correspondingly, we introduce the following subspaces in $\mathbb{R}^n$,

$$
\begin{aligned}
\mathbb{R}^{\sigma_a} &= \{x \in \mathbb{R}^n \mid x_j = 0 \text{ for all } j \notin \sigma_a\} \\
\mathbb{R}^{\sigma_a^c} &= \{x \in \mathbb{R}^n \mid x_j = 0 \text{ for all } j \notin \sigma_a^c\}.
\end{aligned}
$$

Lemma 6.5. *Let $N \subset E_\infty$ be a finite-dimensional smooth linear subspace of $E = \mathbb{R}^n \oplus W$ so that $C \cap N$ is a closed convex cone. If $a \in C \cap N$ is nonzero and generates an extreme ray R in $C \cap N$, then*

$$\dim(N) - 1 \leq \sharp \sigma_a.$$

If, in addition, N is in good position to C, then

$$\dim(N) - 1 = \sharp \sigma_a.$$

Proof. Assume $R = \mathbb{R}^+ \cdot a$ is an extreme ray in $C \cap N$. Abbreviate $\sigma = \sigma_a$ and let σ^c be a complement of σ in $\{1, \ldots, n\}$. Then $R \subset C \cap N \cap (\mathbb{R}^{\sigma^c} \oplus W)$. Let $y \in C \cap N \cap (\mathbb{R}^{\sigma^c} \oplus W)$ be a nonzero element. Since $a_i > 0$ for all $i \in \sigma^c$,

there exists $\lambda > 0$ so that $\lambda a - y \in C \cap N \cap (\mathbb{R}^{\sigma^c} \oplus W) \subset C \cap N$. We conclude $y \in R$ because R is an extreme ray. Given any element $z \in N \cap (\mathbb{R}^{\sigma^c} \oplus W)$ we find $\lambda > 0$ so that $\lambda a + z \in C \cap N \cap (\mathbb{R}^{\sigma^c} \oplus W)$ and infer, by the previous argument, that $\lambda a + z \in R$. This implies that $z \in \mathbb{R}a$. Hence

$$\dim(N \cap (\mathbb{R}^{\sigma^c} \oplus W)) = 1. \tag{28}$$

The projection $p : \mathbb{R}^n \oplus W = \mathbb{R}^\sigma \oplus (\mathbb{R}^{\sigma^c} \oplus W) \to \mathbb{R}^\sigma$ induces a linear map

$$p : N \to \mathbb{R}^\sigma \tag{29}$$

which by (28) has an one-dimensional kernel. Therefore,

$$\sharp\sigma = \dim(\mathbb{R}^\sigma) \geq \dim(N) - 1.$$

Next assume N is in good position to C. Hence there exist a constant $c > 0$ and an sc-complement $N^\perp$ such that $N \oplus N^\perp = \mathbb{R}^n \oplus W$ and if $(n, m) \in N \oplus N^\perp$ satisfies $\|m\| \leq c\|n\|$, then $n + m \in C$ if and only if $n \in C$. We claim that $N^\perp \subset \mathbb{R}^{\sigma^c} \oplus W$. Indeed, let m be any element of $N^\perp$. Multiplying m by a real number we may assume $\|m\| \leq c\|a\|$. Then $a + m \in C$ since $a \in C$. This implies that $m_i \geq 0$ for all indices $i \in \sigma_a$. Replacing m by $-m$, we conclude $m_i = 0$ for all $i \in \sigma_a$. So $N^\perp \subset \mathbb{R}^{\sigma^c} \oplus W$ as claimed. Take $k \in \mathbb{R}^{\sigma_a}$ and write $(k, 0) = n + m \in N \oplus N^\perp$. From $N^\perp \subset \mathbb{R}^{\sigma^c} \oplus W$, we conclude $k = p(n)$. Hence the map p in (29) is surjective and the desired result follows. ■

6.3 Proof of Propositions 6.1

Assume that N is a smooth finite-dimensional subspace of $E = \mathbb{R}^n \oplus W$ in good position to the partial quadrant $C = [0, \infty)^n \oplus W$. Thus, by definition, there is an sc-complement $N^\perp$ of N in E and a constant $c > 0$ so that if $(n, m) \in N \oplus N^\perp$ satisfies $\|m\| \leq c\|n\|$, then the statements $n \in C$ and $n + m \in C$ are equivalent.

We introduce the subset

$$\Sigma = \bigcup_{a \in C \cap N, a \neq 0} \sigma_a \subset \{1, \ldots, n\}.$$

and denote by Σ^c the complement $\{1, \ldots, n\} \setminus \Sigma$. The associated subspaces of $\mathbb{R}^n$ are defined by $\mathbb{R}^\Sigma = \{x \in \mathbb{R}^n \mid x_j = 0 \text{ for } j \notin \Sigma\}$ and $\mathbb{R}^{\Sigma^c} = \{x \in \mathbb{R}^n \mid x_j = 0 \text{ for } j \notin \Sigma^c\}$.

Lemma 6.6. $N^{\perp} \subset \mathbb{R}^{\Sigma^c} \oplus W$.

Proof. Take $m \in N^{\perp}$. We have to show that $m_i = 0$ for all $i \in \Sigma$. So fix an index $i \in \Sigma$ and let a be a nonzero element of $C \cap N$ such that $i \in \sigma_a$. Multiplying a by a suitable positive number we may assume $\|m\| \leq c\|a\|$. Since $a \in C$, we infer that $a + m \in C$. This implies that $a_i + m_i \geq 0$. By definition of σ_a, we have $a_i = 0$ implying $m_i \geq 0$. Replacing m by $-m$ we find $m_i = 0$. Hence $N^{\perp} \subset \mathbb{R}^{\Sigma^c} \oplus W$ as claimed. ∎

Identifying W with $\{0\} \oplus W$ we take an algebraic complement $\widetilde{N}$ of $N \cap W$ in N so that

$$N = \widetilde{N} \oplus (N \cap W) \quad \text{and} \qquad E = \widetilde{N} \oplus (N \cap W) \oplus N^{\perp}. \tag{30}$$

Lemma 6.7. *If the subspace N of E is in good position to the quadrant C, then $\widetilde{N}$ is also in good position to C and the subspace $\widetilde{N}^{\perp} = (N \cap W) \oplus N^{\perp}$ is a good complement of $\widetilde{N}$ in E.*

Proof. Since N is in good position to the quadrant C in E, there exist a constant $c > 0$ and an sc-complement $N^{\perp}$ of N in E such that if $(n, m) \in N \oplus N^{\perp}$ satisfies $\|m\| \leq c\|n\|$, then the statements $n \in C$ and $n + m \in C$ are equivalent. Since E is a Banach space and N is a finite dimensional subspace of E, there is a constant $c_1 > 0$ such that $\|n + m\| \geq c_1[\|n\| + \|m\|]$ for all $(n, m) \in N \oplus N^{\perp}$. To prove that $\widetilde{N}$ is in good position to C, we shall show that $\widetilde{N}^{\perp} := (N \cap W) \oplus N^{\perp}$ is a good complement of $\widetilde{N}$ in E in the sense of Definition 4.10. Let $(\widetilde{n}, \widetilde{m}) \in \widetilde{N} \oplus \widetilde{N}^{\perp} = E$ and assume that $\|\widetilde{m}\| \leq c_1 c\|\widetilde{n}\|$. Write $\widetilde{m} = n_1 + n_2 \in (N \cap W) \oplus N^{\perp}$. Since $c_1[\|n_1\| + \|n_2\|] \leq \|n_1 + n_2\| = \|\widetilde{m}\| \leq c_1 c\|\widetilde{n}\|$, we get $\|n_2\| \leq c\|\widetilde{n}\|$. Note that $\widetilde{n} + \widetilde{m} = \widetilde{n} + n_1 + n_2 \in C$ if and only if $\widetilde{n} + n_2 \in C$ since $n_1 \in \{0\} \oplus W$. Since $\|n_2\| \leq c\|\widetilde{n}\|$, this is equivalent to $\widetilde{n} \in C$. It remains to show that $\widetilde{N} \cap C$ has a nonempty interior. By assumption $N \cap C$ has nonempty interior. Hence there is a point $n \in N \cap C$ and $r > 0$ such that the ball $B_r^N(n)$ in N is contained in $N \cap C$. Write $n = \widetilde{n} + w$ where $\widetilde{n} \in \widetilde{N}$ and $w \in N \cap W$. Since $n \in C$ and $w \in W$, we conclude that $\widetilde{n} \in C$. Hence $\widetilde{n} \in \widetilde{N} \cap C$. Take $\nu \in B_r^{\widetilde{N}}(\widetilde{n})$, the open ball in $\widetilde{N}$ centered at $\widetilde{n}$ and of radius $r > 0$. We want to prove that $\nu \in C$. Since $C = [0, \infty)^n \oplus W \subset \mathbb{R}^n \oplus W$, we have to prove for $\nu = (\nu', \nu'') \in \mathbb{R}^n \oplus W$ that $\nu' \in [0, \infty)^n$. We estimate $\|(\nu + w) - n\| = \|(\nu + w) - (\widetilde{n} + w)\| = \|\nu - \widetilde{n}\| < r$ so that $\nu + w \in B_r^N(n)$ and hence $\nu + w \in N \cap C$. Having identified W with $\{0\} \oplus W$, we have $w = (0, w'') \in \mathbb{R}^n \oplus W$. Consequently,

$\nu + w = (\nu', \nu'' + w'') \in N \cap C$ implies $\nu' \in [0, \infty)^n$. Since also $\nu \in \widetilde{N}$, one concludes that $\nu \in \widetilde{N} \cap C$ and that $\widetilde{n}$ belongs to the interior of $\widetilde{N} \cap C$ in $\widetilde{N}$. The proof of Lemma 6.7 is complete. ■

Since by Lemma 6.6, $N^{\perp} \subset \mathbb{R}^{\Sigma^c} \oplus W$, the good complement $\widetilde{N}^{\perp}$ satisfies

$$\widetilde{N}^{\perp} \subset \mathbb{R}^{\Sigma^c} \oplus W.$$

In particular, $\dim \widetilde{N} = \operatorname{codim} \widetilde{N}^{\perp} \geq \sharp\Sigma$.

Moreover, since C is a closed convex cone in E and $\widetilde{N}$ a subspace of E, we have proved the following lemma.

Lemma 6.8. *The intersection $C \cap \widetilde{N}$ is a closed convex cone in $\widetilde{N}$.*

Using the above lemma and Lemma 6.7, we conclude from Lemma 6.5

$$\dim \widetilde{N} - 1 = \sharp\sigma_a \tag{31}$$

for every generator a of an extreme ray in $C \cap \widetilde{N}$.

The position of $\widetilde{N}$ with respect to $\mathbb{R}^{\Sigma^c} \oplus W$ is as follows.

Lemma 6.9. *Either $\widetilde{N} \cap (\mathbb{R}^{\Sigma^c} \oplus W) = \{0\}$ or $\widetilde{N} \subset \mathbb{R}^{\Sigma^c} \oplus W$. In the second case $\dim \widetilde{N} = 1$ and $\Sigma = \emptyset$.*

Proof. Assume that $\widetilde{N} \cap (\mathbb{R}^{\Sigma^c} \oplus W) \neq \{0\}$. Take a nonzero point $x \in \widetilde{N} \cap (\mathbb{R}^{\Sigma^c} \oplus W)$. We know that $\widetilde{N} \cap C$ has a nonempty interior in $\widetilde{N}$ and is therefore generated as the convex hull of its extreme rays by Lemma 6.3. Let $a \in C \cap \widetilde{N}$ be a generator of an extreme ray R. Then $a_i > 0$ for all $i \in \Sigma^c$ and hence $\lambda a + x \in C \cap \widetilde{N}$ for large $\lambda > 0$. Taking another large number $\mu > 0$, we get $\mu a - (\lambda a + x) \in C \cap \widetilde{N}$. Since $R = \mathbb{R}^+ \cdot a$ is an extreme ray, we conclude $\lambda a + x \in \mathbb{R}^+ \cdot a$ so that $x \in \mathbb{R} \cdot a$. Consequently, there is only one extreme ray in $\widetilde{N} \cap C$, namely $R = \mathbb{R}^+ \cdot a$ with $a \in \mathbb{R}^{\Sigma^c} \oplus W$. Since $\widetilde{N} \cap C$ has a nonempty interior in $\widetilde{N}$, we conclude that $\dim \widetilde{N} = 1$. Hence $\widetilde{N} = \mathbb{R} \cdot a$ and $\widetilde{N} \subset \mathbb{R}^{\Sigma^c} \oplus W$. From equation (31) we also conclude that $a_i > 0$ for all $1 \leq i \leq n$. This in turn implies that $\Gamma = \emptyset$ since $a \in \mathbb{R}^{\Sigma^c} \oplus W$. The proof of Lemma 6.9 is complete. ■

In order to complete the proof of Proposition 6.1, we have to consider, according to Lemma 6.9, two cases. Starting with the first case we assume

that $\widetilde{N} \cap (\mathbb{R}^{\Sigma^c} \oplus W) = \{0\}$. The projection $p : \widetilde{N} \oplus \widetilde{N}^{\perp} = \mathbb{R}^{\Sigma} \oplus (\mathbb{R}^{\Sigma^c} \oplus W) \to \mathbb{R}^{\Sigma}$ induces the linear map

$$p : \widetilde{N} \to \mathbb{R}^{\Sigma}. \tag{32}$$

Take $k \in \mathbb{R}^{\Sigma}$ and write $(k, 0) = n + m \in \widetilde{N} \oplus \widetilde{N}^{\perp}$. Since $\widetilde{N}^{\perp} \subset \mathbb{R}^{\Sigma^c} \oplus W$, we conclude that

$$p(n + m) = p(n) = k$$

so that p is surjective. If $n \in \widetilde{N}$ and $p(n) = 0$, then $n \in \widetilde{N} \cap (\mathbb{R}^{\Sigma^c} \oplus W) = \{0\}$ by assumption. Hence the map in (32) is a bijection. By Lemma 6.4, $C \cap \widetilde{N}$ is a quadrant in $\widetilde{N}$. We shall show that p maps the quadrant $C \cap \widetilde{N}$ onto the standard quadrant $Q^{\Sigma} = [0, \infty)^{\Sigma}$ in $\mathbb{R}^{\Sigma}$. Let a be a nonzero element in $C \cap \widetilde{N}$ generating an extreme ray $R = \mathbb{R}^+ \cdot a$. Then, by Lemma 6.5,

$$\dim \widetilde{N} - 1 = \sharp \sigma_a$$

and since $\sharp \Sigma = \dim \widetilde{N}$ there is exactly one index $i \in \Sigma$ for which $a_i > 0$. Further, $a_i > 0$ for all $i \in \Sigma^c$ by definition of Σ. This implies that there can be at most $\dim(\widetilde{N})$-many extreme rays. Indeed, if a and a' generate extreme rays and $a_i, a'_i > 0$ for some $i \in \Sigma$, then $a_k = a'_k = 0$ for all $k \in \Sigma \setminus \{i\}$. Hence, from $a_j > 0$ for all $j \in \Sigma^c$, we conclude $\lambda a - a' \in C$ for large $\lambda > 0$. Therefore, $a' \in \mathbb{R}^+ a$ implying that a and a' generate the same extreme ray. As a consequence, $\widetilde{N} \cap C$ has precisely $\dim \widetilde{N}$-many extreme rays because $\widetilde{N} \cap C$ has a nonempty interior in view of Lemma 6.7. Hence the map p in (32) induces an isomorphism

$$(\widetilde{N}, C \cap \widetilde{N}) \to (\mathbb{R}^{\Sigma}, Q^{\Sigma}).$$

This implies that $(N, C \cap N)$ is isomorphic to $\left(\mathbb{R}^{\dim(N)}, [0, \infty)^{\sharp\Sigma} \oplus \mathbb{R}^{\dim(N) - \sharp\Sigma}\right)$.

In the second case we assume that $\widetilde{N} \subset \mathbb{R}^{\Sigma^c} \oplus W$. From Lemma 6.9, $\Sigma = \emptyset$ and $\widetilde{N} = \mathbb{R} \cdot a$ for an element $a \in C \cap \widetilde{N}$ satisfying $a_i > 0$ for all $1 \leq i \in\leq n$. Hence $(\widetilde{N}, \widetilde{N} \cap C)$ is isomorphic to $(\mathbb{R}, \mathbb{R}^+)$ and therefore $(N, C \cap N)$ is isomorphic to $(\mathbb{R}, \mathbb{R}^+)$ since in this case $N = \widetilde{N}$. The proof of Proposition 6.1 is complete. ■

6.4 Determinants and Orientation.

We shall outline the definition of determinant bundles associated to families of linearized polyfold Fredholm operators. While such constructions are

well-known in the classical case there are some subtleties in our case. The essential problem is that the linearisations do not depend as operators continuously on the points where they were linearized. Nevertheless what makes the construction possible is the fact that a Fredholm section has the contraction germ property. We shall discuss the construction in more detail in [18] and just give some outline in the following. We recall some facts about determinants of linear Fredholm operators. For more details we refer to [3] and [7]. If A is a finite-dimensional real vector space we denote by $\wedge^{\max}A$ its maximal (nontrivial) wedge. In case $A=\{0\}$ we put $\wedge^{\max}A=\mathbb{R}$. We begin with the linear algebra fact that given an exact sequence

$$0\rightarrow A_0\rightarrow A_1\rightarrow A_2\rightarrow\cdots\rightarrow A_n\rightarrow 0$$

between vector spaces, there is a natural isomorphism (constructed from the maps in the exact sequence by formulae depending smoothly on the ingredients)

$$\otimes_{i\ \text{even}}(\wedge^{\max}A_i)\rightarrow\otimes_{i\ \text{odd}}(\wedge^{\max}A_i).$$

There exist also natural isomorphisms $A\otimes B\rightarrow B\otimes A$, $(\wedge^{\max}E)\otimes(\wedge^{\max}E)^*\rightarrow\mathbb{R}$ and $\wedge^{\max}(E^*)\rightarrow(\wedge^{\max}E)^*$ where $*$ refers to the dual space. Of course, there are natural isomorphisms $A\otimes\mathbb{R}\rightarrow A$.

Assume that $T:E\rightarrow F$ is a linear Fredholm operator between Banach spaces. Then we define its **determinant** $\det(T)$ as the one-dimensional real vector space

$$\det(T)=(\wedge^{\max}\ker(T))\otimes(\wedge^{\max}\mathrm{coker}(T))^*.$$

An orientation of the Fredholm operator T is by definition an orientation of the vector space $\det(T)$. An orientation of $\det(T)$ can be given by a pair of orientations for $\wedge^{\max}\ker(T)$ and $\wedge^{\max}\mathrm{coker}(T)$. Of course, reversing both orientations defines the same orientation for T. If T is an isomorphism, then $\det(T)=\mathbb{R}\otimes\mathbb{R}^*$ which is naturally isomorphic to $\mathbb{R}$ by the map $e\otimes e^*\mapsto e^*(e)$. In this case 1 orients T. We call it the **natural orientation of an isomorphism**. If $T:E\rightarrow F$ is a surjective Fredholm operator, then $\det(T)=(\wedge^{\max}\ker(T))\otimes\mathbb{R}^*$. In this case an orientation $\mathfrak{o}$ of $\det(T)$ determines an orientation of $\ker(T)$ by requiring that, paired with the canonical orientation of $\mathbb{R}^*$ by 1^*, it gives the orientation of T. Hence an orientation for a surjective Fredholm operator can be viewed as being equivalent to an orientation of $\ker(T)$. Since $T:E\rightarrow F$ is Fredholm, we can

take a projection $P : F \to F$ so that the range $R(P)$ has finite codimension and $PT : E \to P(F)$ is surjective. From the exact sequence of maps

$$0 \to \ker(T) \to \ker(PT) \xrightarrow{T} (I-P)F \to F/R(T) \to 0$$

we derive the natural isomorphism

$$(\wedge^{\max} \ker(T)) \otimes (\wedge^{\max}(I-P)F) \to (\wedge^{\max} \ker(PT)) \otimes (\wedge^{\max} \mathrm{coker}(T)).$$

Tensoring with $(\wedge^{\max}(I-P)F)^*$ from the right we obtain the natural isomorphism

$$\begin{aligned}\wedge^{\max} \ker(T) &\to (\wedge^{\max} \ker(PT)) \otimes (\wedge^{\max}\mathrm{coker}(T)) \otimes (\wedge^{\max}(I-P)F)^* \\ &\to (\wedge^{\max} \ker(PT)) \otimes (\wedge^{\max}(I-P)F)^* \otimes (\wedge^{\max}\mathrm{coker}(T)).\end{aligned}$$

By tensoring from the right with $(\wedge^{\max}\mathrm{coker}(T))^*$ we finally end up with the natural isomorphism

$$\det(T) \to \det(PT).$$

Next we consider a continuous (in the operator topology) arc of Fredholm operators $t \mapsto T_t$ for $t \in [0,1]$. We claim that there is a projection P so that the operator $PT_t : E \to PF$ is surjective for every $0 \leq t \leq 1$.

To prove the claim, take $0 \leq t^* \leq 1$. The Banach spaces E and F split as follows $E = E' \oplus \ker(T_{t^*})$ and $F = C_{t^*} \oplus R(T_{t^*})$ where $C_{t^*} = \mathrm{coker}\ T_{t^*}$ is the cokernel of T_{t^*}. Denoting by Q the projection $Q : C_{t^*} \oplus R(T_{t^*}) \to R(T_{t^*})$, the map $QT_{t^*}|E' : E' \to R(T_{t^*})$ is an isomorphism and we find an open interval $I(t^*)$ around t^* in $[0,1]$ such that $QT_t|E' : E' \to R(T_{t^*})$ is an isomorphism for $t \in I(t^*)$. Hence $QT_t : E \to R(T_{t^*})$ is a surjection for $t \in I(t^*)$. Now we cover $[0,1]$ by a finite number of such intervals $I(t_1), \ldots, I(t_n)$ and denote by $Q_i : C_i \oplus R_i \to R_i$ the corresponding projections onto $R_i = R(Q_i) = R(T_{t_i})$ so that operators $Q_iT_t : E \to R_i$ are surjective for every $t \in I(t_i)$. With $C = \mathrm{span}\{C_1, \ldots, C_n\}$, there exists a topological complement R of C so that $F = C \oplus R$. Denoting by $P : C \oplus R \to R$ the projection onto E along C, we claim that $PT_t : E \to R$ is a surjection for every $t \in [0,1]$. Indeed, fix $t \in [0,1]$ and choose $y \in R$. Then $t \in I(t_i)$ for some i and $y = a + b$ where $a \in C_i$ and $b \in R_i$. Hence there exists a point $x \in E$ solving $Q_iT_t(x) = b$. This implies that $T_t(x) = a' + b$ with $a' \in C_i$. Hence $T_t(x) = (a' - a) + y$ and since $(a' - a) \in C_i \subset C$, it follows that $PT_t(x) = y$. Consequently, $PT_t : E \to P(F)$ is surjective for every $0 \leq t \leq 1$ as claimed.

Proceeding as before we obtain a natural family of isomorphisms

$$\varphi_t : \det(T_t) \to \det(PT_t).$$

Now we observe that the family $t \mapsto PT_t$ is a continuous arc of surjective Fredholm operators so that $\dim \ker PT_t$ is constant. Consequently, the family defines in a natural way a topological line bundle over $[0,1]$ by

$$L = \bigcup_{t\in[0,1]} \{t\} \times (\wedge^{\max} \ker(PT_t)) \otimes (\wedge^{\max}(I-P)F)^*.$$

We have a natural bijection of the line bundle L to the line bundle

$$K = \bigcup_{t\in[0,1]} \{t\} \times (\wedge^{\max} \ker(T_t)) \otimes (\wedge^{\max}\mathrm{coker}(T_t))^*.$$

It is linear in the fibers. The punch-line is that taking another projection Q so that still $QT_t : E \to R(Q)$ is surjective we obtain another topological line bundle L' and the transition map $L \to L'$ is a topological bundle isomorphism. This implies that K carries in a natural way the structure of a topological line bundle. This fact is a priori not obvious since the kernel and cokernel dimensions vary. It is our aim to carry these ideas over to the polyfold framework. One has to be somewhat careful since our notion of differentiability is so weak. For example, we might have a section f whose linearisations $f'(x(t))$ along a path $x(t)$ are all Fredholm, but these operators need not be continuously depending on $x(t)$ as linear operators. As we will see it will nevertheless be possible to carry out the above ideas.

Assume we are given a fillable M-polyfold bundle $p : E \to X$ and a Fredholm section f. From now on we assume everything is built on separable sc-Hilbert spaces. We want to introduce the notion of an orientation $\mathfrak{o}$ for f. Assume that $x \in X$ is a smooth point, i.e., $x \in X_\infty$. Then we can look at the set of all linearisations of f at x. Any two such linearisation differ by an sc^+-operator. Every loop of linearisations is contractible. Hence if we have oriented one linearisation, then there is a natural orientation for all the other linearisations. Therefore, we can talk at the point x of an orientation for the linearisation.

Next we study the question of continuation. Assume that $\gamma : [0,1] \to X_\infty$ is an sc^∞-path connecting $x = \gamma(0)$ with $y = \gamma(1)$. We find an sc-smooth sc^+-map $s : [0,1] \times X \to E$ so that $f(\gamma(t)) + s(t,\gamma(t)) = 0$ for all $0 \leq t \leq 1$.

Fix $t_0 \in (0,1)$ (the cases $t_0 = 0$ or $t_0 = 1$ are done in a similar way) and consider the linearisations at $\gamma(t_0)$,

$$(f + s_{t_0})'(\gamma(t_0)) : T_{\gamma(t_0)}X \to E_{\gamma(t_0)}.$$

We consider f as a section of the bundle $Y \to [0,1] \times X$. Locally around the point $(t_0, \gamma(t_0)) \in [0,1] \times X$, the section f has a filler and so the section $f + s$ also has a filler. It can be chosen in such a way that in suitable strong bundle coordinates we have a contraction germ. The linearisation of a filled section is a stabilization of the linearized unfilled section by an isomorphism of the complements of the fibers. Hence the determinant $\det((f + s_t)'(\gamma(t)))$ for t close to t_0 can be identified using the local coordinates with that of the filled object. Therefore, we may assume that the section $f + s$ is already filled and has the contraction normal form. More precisely we assume that the section $f + s$ is of the form

$$f + s : O \subset (\mathbb{R} \times [0,\infty)^k \oplus R^{n-k}) \oplus W \to \mathbb{R}^N \oplus W$$

where O is a relatively open neighborhood of $(t_0, 0, 0)$ which corresponds to $(t_0, \gamma(t_0))$. With the projection $P : \mathbb{R}^N \oplus W \to W$, the map

$$P(f + s)(t, a, b) = b - B(t, a, b),$$

in which $a \in [0,\infty)^k \oplus \mathbb{R}^{n-k}$ and $b \in W$, has the contraction germ property near $(t_0, 0, 0)$.

Denote by $(a(t), b(t)) \in [0,\infty)^k \oplus R^{n-k}) \oplus W$ a point which corresponds in our local coordinates to $\gamma(t)$ for t close to t_0. Keeping t fixed and linearizing the above map at the point $(t, a(t), b(t))$ with respect to the variable (a, b) we find that

$$P(f + s_t)'(a(t), b(t)) = 1 - D_2(t, a(t), b(t)) - D_3B(t, a(t), b(t))$$

where 1 stands for the identity map $W \to W$. In view of the proof of Theorem 2.3, the linear map $1 - D_3B(t, a(t), b(t)) : W \to W$ is an isomorphism so that the linearization $P(f + s_t)'(a(t), b(t)) : \mathbb{R}^n \oplus W \to W$ is a surjection. Consequently, we obtain a family of surjective sc-Fredholm operators

$$P(f + s_t)'(a(t), b(t)) : \mathbb{R}^n \oplus W \to W,$$

for t close to t_0. Moreover, the kernel is changing smoothly with t near t_0. This implies that the associated locally defined determinant bundle is locally

a topological line bundle which, by the discussion above and rolling back the coordinate changes, implies that the original family $t \mapsto \det(f+s_t)'(\gamma(t))$ defines a topological line bundle

$$L(f)_\gamma := \bigcup_{t\in[0,1]} \{t\} \times \det(f+s_t)'(\gamma(t))$$

in a natural way.

Two orientations $\mathfrak{o}_x$ and $\mathfrak{o}_y$ for the linearisation of f at the points x and y, respectively, are called **related by continuation** along an sc-smooth path connecting x and y if the associated topological line bundle $L(f)_\gamma$ can be oriented in such a way that it induces the given orientations at the ends.

Definition 6.10. *Let f be a Fredholm section of the fillable strong M-polyfold bundle $p : E \to X$ whose local models are built on separable sc-Hilbert spaces. Then f is called* **orientable** *if at every smooth point the linearizations can be oriented in such a way that they are related by continuation along arbitrary pathes. If f is orientable a coherent choice of orientations $x \to \mathfrak{o}_x$ of the linearisations of f at x is called an* **orientation** *for f. We write $(f, \mathfrak{o})$ for an* **oriented Fredholm section**.

7 Glossary

In this section we recall some of the basic notions from [12].

- **sc-structure**. An sc-structure on the Banach space E is a nested sequence

  $$E = E_0 \supset E_1 \supset E_2 \cdots \supset E_\infty = \bigcap_{k\geq 0} E_k$$

  of Banach spaces E_m, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, having the following properties.

  (1) If $m < n$, the inclusion $E_n \to E_m$ is a compact operator.

  (2) The vector space E_∞ is dense in E_m for every $m \geq 0$.

  A Banach space E equipped with an sc-structure (E_m) is called **sc-smooth**. Each of the spaces E_m is an sc-Banach space and denoted by E^m. The sc-structure on E^m is given by $(E_{m+k})_{k\geq 0}$. Points and sets contained in E_∞ are called **smooth points** and **smooth sets**.

- **direct sum and $\triangleleft$-sum** If $(E_n)_{n\geq 0}$ and $(F_n)_{n\geq 0}$ are sc-smooth structures of E and F, then $E \oplus F$ carries the sc-structure defined by $(E \oplus F)_n = E_n \oplus F_n$. By $E \triangleleft F$ we denote the Banach space $E \oplus F$ equipped with the bi-filtration $(E \triangleleft F)_{m,k} = E_m \oplus F_k$ for pairs (m, k) satisfying $m \geq 0$ and $0 \leq k \leq m + 1$.

- **sc^0-map**. A map $\varphi : U \to V$ between open subsets of sc-Banach spaces is said to be class sc^0 or sc^0, if $\varphi(U_m) \subset V_m$ and the induced maps $\varphi : U_m \to V_m$ are continuous for all $m \geq 0$.

- **sc-operator**. A bounded linear operator $T : E \to F$ between sc-Banach spaces is called an **sc-operator** if T is of class sc^0. If, in addition, T is bijective and $T^{-1} : F \to E$ is sc^0, then T is called an **sc-isomorphism**.

- **partial quadrant**. Given an sc-Banach space W, a subset $C \subset W$ is a partial quadrant of W if there is an sc-Banach space Q and an sc-isomorphism $T : W \to \mathbb{R}^n \oplus Q$ such that $T(C) = [0, \infty)^n \oplus Q$.

- **induced sc-structure** If U is a relatively open subset of a partial quadrant C in the sc-Banach space E, then the nested sequence of sets $U_m = U \cap E_m$ is called the **induced sc-structure** on U. The set U_m inherits sc-smooth structure defined by $(U_{m+k})_{k\geq 0}$. The set U_m equipped with this induced sc-structure is denoted by U^m.

- **tangent bundle TU**. Given a relatively open subset U of the partial quadrant C in the sc-Banach space E, the tangent bundle TU of U is defined as $TU = U^1 \oplus E$. That is, the sc-smooth structure of TU, is given by the nested sequence $(TU)_m := U_{m+1} \oplus E_m$ for all $m \geq 0$. The canonical projection $p : TU \to U^1$ is of class sc^0. The higher order tangent bundles T^kU are defined iteratively as $T^1U = TU$ and $T^kU = T(T^{k-1}U)$ for $k \geq 2$.

- **sc-subspace**. If E is an sc-Banach space, then a closed subspace F of E is called sc-subspace of E if the nested sequence $F_m = F \cap E_m$ is an sc-structure for F. An sc-subspace $F \subset E$ **splits** E if there exists another sc-subspace G of E so that $E_m = F_m \oplus G_m$ for all $m \geq 0$.

- **Fredholm operator**. An sc-operator $T : E \to Y$ is called Fredholm provided that there are sc-splittings $E = K \oplus X$ and $F = Y \oplus C$ having the following properties.

(1) $K = \text{kernel}(T)$ is finite dimensional.

(2) C is finite dimensional.

(3) $Y = T(X)$ and $T : X \to Y$ is an sc-isomorphism.

The finite dimensional vector spaces $K \subset E$ and $C \subset F$ are smooth.

- **sc$^+$-operator**. An sc-operator $T : E \to F$ is called an sc$^+$-operator if $T(E_m) \subset E_{m+1}$ for every $m \geq 0$ and $T : E \to E^1$ is of class sc^0.

- **sc^1-map** Let E, F be sc-Banach spaces and let U be a relatively open subset of a partial quadrant C contained in the sc-Banach space E. An sc^0-map $f : U \to F$ is said to be **sc^1** or of **class sc^1** if the following holds.

  (1) For every $x \in U_1$, there exists a bounded linear map $Df(x) \in \mathcal{L}(E_0, F_0)$ satisfying (with $x + h \in U_1$)

  $$\frac{1}{\|h\|_1}\|f(x+h) - f(x) - Df(x)h\|_0 \to 0 \quad \text{as } \|h\|_1 \to 0.$$

  (2) The **tangent map** $Tf : TU \to TF$, defined by

  $$Tf(x, h) = (f(x), Df(x)h)$$

  is an sc^0-map.

- **sc^k-map**. Let U be a relatively open subset of a partial quadrant C contained in the sc-Banach space E and let F be another sc-Banach space. A map $f : U \subset E \to F$ is an sc^k-map or of class sc^k if the sc^0-map $T^{k-1}f : T^{k-1}U \to T^{k-1}F$ is of class sc^1. In this case the tangent map $T(T^{k-1}f) : T(T^{k-1}U) \to T(T^{k-1}F)$ is denoted by $T^k f$. If $f : U \subset E \to F$ is of class sc^k for every $k \geq 0$, then it is called **sc-smooth** or of **class sc$^\infty$**.

- **sc-diffeomorphism**. A homeomorohism $f : U \to V$ between relatively open subsets of partial quadrants in sc-Banach spaces is called sc-diffeomorphism if f and f^{-1} are sc-smooth.

- **sc-smooth splicing**. Let V be an open subset of a partial quadrant $C \subset W$, let E be an sc-Banach space and let $\pi_v : E \to E$ be a bounded linear projection for every $v \in V$ such that the map

$$\pi : V \oplus E \to E, \quad (v,e) \mapsto \pi_v(e)$$

is sc-smooth. Then the triple $\mathcal{S} = (\pi, E, V)$ is called an **sc-smooth splicing**.

- **splicing core**. Let $\mathcal{S} = (\pi, E, V)$ be an sc-smooth splicing. The associated splicing core is the subset of $V \oplus E$ defined by

$$K^{\mathcal{S}} = \{(v,e) \in V \oplus E |\, \pi_v(e) = e\}.$$

- **tangent splicing of $\mathcal{S}$**. Given a splicing $\mathcal{S} = (\pi, E, V)$, the tangent splicing of $\mathcal{S}$ is the triple defined by

$$T\mathcal{S} = (T\pi, TE, TV).$$

- The **splicing core of the tangent splicing** $T\mathcal{S}$ is the set

$$K^{T\mathcal{S}} = \{(v, \delta v, e, \delta e) \in TV \oplus TE |\, (T\pi)_{(v,\delta v)}(e, \delta e) = (e, \delta e)\}.$$

- A **local M-polyfold model** consists of a pair $(O, \mathcal{S})$ in which O is an open subset of the splicing core $K^{\mathcal{S}}$ associated with the sc-smooth splicing $\mathcal{S} = (\pi, E, V)$. The **tangent of the local M-polyfold model** $(O, \mathcal{S})$ is the object defined by

$$T(O, \mathcal{S}) = (K^{T\mathcal{S}} | O^1, T\mathcal{S})$$

where $K^{T\mathcal{S}}|O^1$ denotes the collection of all points in $K^{T\mathcal{S}}$ which project under the canonical projection $K^{T\mathcal{S}} \to (K^{\mathcal{S}})^1$ onto the points in O^1.

- **smooth maps between splicing cores**. Given open subsets O, O' of splicing cores $K^{\mathcal{S}} \subset V \oplus E$ and $K^{\mathcal{S}'} \subset V' \oplus E'$ where $V$and V' are open subsets of partial quadrants in the sc-Banach spaces W and W', define the open set $\widehat{O} \subset V \oplus E$ by $\widehat{O} = \{(v,e) \in V \oplus E | (v, \pi_v(e)) \in O\}$. An sc^0-map $f : O \to O'$ is of class sc^1 provided the map

$$\widehat{f} : \widehat{O} \subset V \oplus E \to W' \oplus E', \quad \widehat{f}(v,e) = f(v, \pi_v(e))$$

is of class sc^1. The tangent map $T\widehat{f}$ associated with the sc^1-map $\widehat{f}$ satisfies $T\widehat{f}(K^{T\mathcal{S}}|O^1) \subset K^{T\mathcal{S}'}|O'$ and induces a map $TO \to TO'$ which is denoted by Tf and called the **tangent map** of f. The tangents TO and TO' are open subsets of the splicing cores $K^{T\mathcal{S}}$ and $K^{T\mathcal{S}'}$, and the notion of f to be of **class sc**^{k} is defined iteratively.

- **M-poyfold**. Let X be a second countable Hausdorff space. An **M-polyfold chart** for X is a triple $(U, \varphi, \mathcal{S})$ in which U is an open subset of X, $\mathcal{S} = (\pi, E, V)$ is an sc-smooth splicing and $\varphi : U \to K^{\mathcal{S}}$ is a homeomorphism onto an open subset of the splicing core $K^{\mathcal{S}} = \{(v,e) \in V \oplus E |\, \pi_v(e) = 0\}$. Two such charts are compatible if the transition maps between open subsets of splicing cores are sc-smooth. A maximal atlas of sc-smoothly compatible M-poyfold charts is called an **M-polyfold structure** on X, and X equipped with such a structure is called **M-polyfold of type 0**. By definition, an M-polyfold looks locally like an open subset of a splicing core.

- **sc-smooth map between M-polyfolds**. A map $f : X \to X'$ is called of class sc^0, resp. sc^k or sc-smooth if for every point $x \in X$ there exist a chart $(U, \varphi, \mathcal{S})$ around x and a chart $(U', \varphi', \mathcal{S}')$ around $f(x)$ so that $f(U) \subset U'$ and
$$\varphi' \circ f \circ \varphi(U) \to \varphi'(U')$$
is of class sc^0, resp. sc^k or sc-smooth.

- A **general sc-smooth splicing** is a triple $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ in which $(O, \mathcal{S})$ is a local M-polyfold model associated with the sc-smooth splicing $\mathcal{S} = (\pi, E, V)$ and O is an open subset of the splicing core $K^{\mathcal{S}} = \{(v,e) \in V \oplus E |\, \pi_v(e) = e\}$. The space F is an sc-Banach space and the map
$$\rho : O \oplus F \to F, \quad ((v,e),u) \mapsto \rho(v,e,u)$$
is sc-smooth. Moreover, for every $(v,e) \in O$, the map $\rho_{(v,e)} = \rho(v,e,\cdot) : F \to F$ is a bounded linear projection. A second countable Hausdorff space equipped with a maximal atlas where the local models are open subsets of general splicings are called **M-polyfolds of type 1**.

- The **tangent of a general splicing** $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ is the triple
$$T\mathcal{R} = (T\rho, TF, (TO, T\mathcal{S})).$$

- A **strong bundle splicing** is a general sc-smooth splicing

$$\mathcal{R} = (\rho, F, (O, \mathcal{S}))$$

having the following **additional property**. If $(v, e) \in O_m$ and $u \in F_{m+1}$, then $\rho((v, e), u) \in F_{m+1}$, and the triple $\mathcal{R}^1 = (\rho, F^1, (O, \mathcal{S}))$ is also a general sc-smooth splicing. The **complementary strong bundle splicing** $\mathcal{R}^{\mathrm{c}}$ is defined by $\mathcal{R}^{\mathrm{c}} = (1 - \rho, F, (O, \mathcal{S}))$.

- **splicing core of the strong bundle splicing**. Given a strong bundle splicing $\mathcal{R} = (\rho, F, (O, \mathcal{S})$, the set

$$K^{\mathcal{R}} = \{(w, u) \in O \oplus F |\, \rho(w, u) = u\}$$

is called the splicing core of the strong bundle splicing $\mathcal{R}$. The splicing core $K^{\mathcal{R}}$ has the bi-filtration

$$K^{\mathcal{R}}_{m,k} = \{(w, u) \in K^{\mathcal{R}} |\, w \in O_m, u \in F_k\}$$

where $m \geq 0$ and $0 \leq k \leq m + 1$. The splicing core $K^{\mathcal{R}}$ can be viewed as a subset of $O \triangleleft F$ and its bi-filtration is the induced one. The bundle $K^{\mathcal{R}} \to O$ is called a **local strong bundle**. With the strong bundle splicing $\mathcal{R}$ there are associated two splicing cores $K^{\mathcal{R}^0}$ and $K^{\mathcal{R}^1}$, denoted by $K^{\mathcal{R}}(0)$ and $K^{\mathcal{R}}(1)$, and equipped with the filtrations

$$K^{\mathcal{R}}(0)_m = K^{\mathcal{R}}_{m,m} \quad \text{and} \quad K^{\mathcal{R}}(1)_m = K^{\mathcal{R}}_{m,m+1}$$

for $m \geq 0$. A **special local strong bundle** is associated with the special strong bundle splicing $\mathcal{R} = (\mathrm{id}, F, (O, \mathcal{S}))$, where $O \subset K^{\mathcal{S}}$ is an open set in the splicing core associated with the splicing $\mathcal{S} = (\mathrm{id}, E, V)$. The special local strong bundle is then given by

$$K^{\mathcal{R}} = O \triangleleft F \to O$$

with the filtrations $K^{\mathcal{R}}(0)_m = O_m \oplus F_m$ and $K^{\mathcal{R}}(1)_m = O_m \oplus F_{m+1}$. We can view O as a local model of an sc-manifold and $O \triangleleft F$ as a model of a local sc-bundle having the base O.

- **$\mathbf{sc}^1_{\triangleleft}$-maps**. Let $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ and $\mathcal{R}' = (\rho', F', (O', \mathcal{S}'))$ be general sc-smooth splicings with associated splicing cores $K^{\mathcal{R}} \subset O \oplus F$ and $K^{\mathcal{R}'} \subset O' \oplus F'$. Let the bundle map $f : K^{\mathcal{R}} \to K^{\mathcal{R}'}$ be of the form

$$f(w, u) = (\varphi(w), \Phi(w, u))$$

where $\varphi : O \to O'$ and $\Phi : O \oplus F \to F'$. Then

(1) f is of **class $\mathbf{sc}^0_\triangleleft$** if it induces sc^0-maps $K^{\mathcal{R}}(i) \to K^{\mathcal{R}'}(i)$ for $i = 0$ and $i = 1$.

(2) f is of **class $\mathbf{sc}^1_\triangleleft$** it it is $\mathrm{sc}^0_\triangleleft$ and induces sc^1-maps $K^{\mathcal{R}}(i) \to K^{\mathcal{R}'}(i)$ for $i = 0$ and $i = 1$.

If $f : K^{\mathcal{R}} \to K^{\mathcal{R}'}$ is a map of class $\mathrm{sc}^1_\triangleleft$, then tangent map $Tf : TK^{\mathcal{R}} \to TK^{\mathcal{R}'}$ is of class $\mathrm{sc}^0_\triangleleft$. If the tangent map Tf is of class $\mathrm{sc}^1_\triangleleft$, then f is said to be of **class $\mathbf{sc}^2_\triangleleft$**. The $\mathrm{sc}^k_\triangleleft$-classes are defined inductively. The map $f : K^{\mathcal{R}} \to K^{\mathcal{R}'}$ is of **class $\mathbf{sc}^\infty_\triangleleft$** or $\mathrm{sc}_\triangleleft$-smooth if it is of class $\mathrm{sc}^k_\triangleleft$ for every k.

The map $f : K^{\mathcal{R}} \to K^{\mathcal{R}'}$ as above is called a **strong bundle map of class $\mathbf{sc}^0_\triangleleft$** if it induces sc^0-maps $K^{\mathcal{R}}(i) \to K^{\mathcal{R}'}(i)$ for $i = 0$ and $i = 1$. It is called a **strong bundle map of class $\mathbf{sc}^1_\triangleleft$** if it is of class $\mathrm{sc}^0_\triangleleft$ and induces sc^1-maps $K^{\mathcal{R}}(i) \to K^{\mathcal{R}'}(i)$ between the splicing cores of the splicings $\mathcal{R}$ and $\mathcal{R}'$, for $i = 0$ and $i = 1$. Proceeding inductively one defines strong bundle maps of class $\mathrm{sc}^\infty_\triangleleft$.

- **sc-smooth section of a local strong bundle** $p : K^{\mathcal{R}} \to O$. Given a a local strong bundle $p : K^{\mathcal{R}} \to O$, a section f of p is called sc-smooth, if f is an sc-smooth section of the bundle $K^{\mathcal{R}}(0) \to O$. The section f is called an **$\mathbf{sc}^+$-smooth section**, if it defines an sc-smooth section of the bundle $K^{\mathcal{R}}(1) \to O$.

- **strong M-polyfold bundle**. Let Y be an M-polyfold of type 1, let X an M-polyfold of type 0, and let $p : Y \to X$ be a surjective sc-smooth map. It is assumed that each fiber $p^{-1}(x) = Y_x$ is a Banach space. A **strong M-polyfold bundle chart** for the $p : Y \to X$ is a triple $(U, \Phi, (K^{\mathcal{R}}, \mathcal{R}))$ in which $U \subset X$ is an open set and $\mathcal{R} = (\rho, F, (O, \mathcal{S}))$ a strong bundle splicing with the local model $(O, \mathcal{S})$ of the M-polyfold X. The map Φ is an sc-diffeomorphism $p^{-1}(U) \to K^{\mathcal{R}}$ which is linear on the fibers and which covers the sc-diffeomorphism $\varphi : U \to O$ so that the following diagram commutes,

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\Phi} & K^{\mathcal{R}} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U' & \xrightarrow{\varphi} & O. \end{array}$$

Moreover, Φ resp. φ are smoothly compatible with the M-polyfold structures on Y and X, respectively.
Two M-polyfold bundle charts $(U, \Phi, (K^{\mathcal{R}}, \mathcal{R}))$ and $(U', \Psi, (K^{\mathcal{R}'}, \mathcal{R}'))$ with Φ covering the sc-diffeomorphism $\varphi : U \to O$ and Ψ covering the sc-diffeomorphism $\psi : U' \to O'$ are $\mathrm{sc}_{\triangleleft}$-compatible if the the transition map

$$\Psi \circ \Phi^{-1} : K^{\mathcal{R}}|\varphi(U \cap U') \to K^{\mathcal{R}'}|\psi(U \cap U')$$

between their splicing cores $K^{\mathcal{R}}$ and $K^{\mathcal{R}'}$ are $\mathrm{sc}_{\triangleleft}$-smooth.
An **M-polyfold bundle atlas** consists of a family of M-polyfold bundle charts $(U, \Phi, (K^{\mathcal{R}}, \mathcal{R}))$ so that the underlying open sets cover X and so that any two charts are $\mathrm{sc}_{\triangleleft}$-compatible. A maximal atlas of M-polyfold bundle charts is called an M-polyfold bundle structure and the map

$$p : Y \to X$$

is called a **strong M-polyfold bundle**.

- **sc-smooth section**. Given a strong M-polyfold bundle $p : Y \to X$, a section $f : X \to Y$ is called sc-smooth, if its local representations in the strong M-polyfold bundle charts are sc-smooth. It is called an **sc$^+$-smooth section** if its local representations in the strong M-polyfold bundle charts are sc$^+$-smooth sections.

- **linearization of an sc-smooth section**. Given a strong M-polyfold bundle $p : Y \to X$ and an sc-smooth section $f : X \to Y$. If $q \in X$ is a smooth point at which the section f vanishes, the linearization of f at q is defined by

$$f'(q) : T_qX \to Y_q, \quad h \mapsto P_q \circ Tf(q)h$$

where P_q is the projection $T_qX \oplus Y_q \to Y_q$. If at the smooth point $q \in X$ the section does not vanish, then the linearization of f at q is defined as follows. Take any sc$^+$-section defined near q satisfying $s(q) = f(q)$. Then the section $f - s$ vanishes at the smooth point q and the **linearization of f with respect to** s is defined by

$$f'_{[s]}(q) : T_qX \to Y_q, \quad h \mapsto P_q \circ T(f - s)(q)h.$$

If s and t are two sc$^+$-sections such that $s(q) = t(q) = f(q)$, then the linearizations $f'_{[s]}(q)$ and $f'_{[t]}(q)$ differ by an sc$^+$-operator. In particular,

if one linearization is an sc-Fredholm operator, then also the other linearization is an sc-Fredholm operator having the same Fredholm index in view of Proposition 2.11 in [12].

- **linearized Fredholm section**. An sc-smooth section f of the strong M-polyfold bundle $p : Y \to X$ is called linearized Fredholm at the smooth point $q \in X$ if the linearization of f at q is an sc-Fredholm operator. The section is called linearized Fredholm, if this holds true at all smooth points q.

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